**Question**

The height of the ground in kilometers near an extinct volcano is given by the formula :

$$h = \exp -(x^2 + y^2 - 0.25)^2$$

where $x$ and $y$ are the distances in kilometers from the centre of the crater in the north and east directions respectively.

**(a)** Sketch the shape of the volcano in section; find the height of the centre of the crater, of the rim around the crater, and at a large distance from the mountain.

Now find the slope of the paths:

**(b)** at $(0.5, 0)$ in the $(1, 0)$ direction

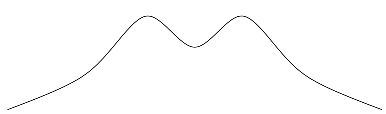
**(c)** at $(1, 1)$ in the $(1, 1)$ direction

**(d)** at $(1, 1)$ in the $(2, -1)$ direction

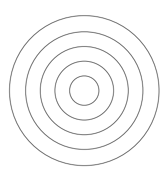
**Answer**

$$h = \exp -(x^2 + y^2 - 0.25)^2 = \exp -f(xy)$$

**(a)** Shape:



cross − section



contours

At the centre of the crater $x = y = 0$ so $h = \exp -0.25^2 \approx 0.9394$km

On the rim $x^2 + y^2 = 0.25$; rim has radius $\frac{1}{2}$km and $h = 1$

(the crater is approximately 60.6 metres below the rim)

Far from volcano $x, y \to \infty \Rightarrow h \to 0$

**(b), (c), (d)** all need $\nabla h$

$$\nabla h = \frac{\partial h}{\partial x}\mathbf{i} + \frac{\partial h}{\partial y}\mathbf{j} = \frac{\partial h}{\partial f}\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial h}{\partial f}\frac{\partial f}{\partial y}\mathbf{j}$$

$$\begin{aligned}
f(x,y) &= (x^2+y^2-0.25)^2 \\
\frac{\partial f}{\partial x} &= 2(x^2+y^2-0.25)\times 2x = 4x(x^2+y^2-0.25) \\
\frac{\partial f}{\partial y} &= 2(x^2+y^2-0.25)\times 2y = 4y(x^2+y^2-0.25) \\
h &= \exp(-f) \Rightarrow \frac{dh}{df} = -\exp(-f) = -h
\end{aligned}$$

$$\nabla h = -\exp-(x^2+y^2-0.25)^2(4x(x^2+y^2-0.25)\mathbf{i}+4y(x^2+y^2-0.25)\mathbf{j})$$

**(b)** At $(0.5, 0)$ $\nabla h = -(1)(0\mathbf{i}+0\mathbf{j}) = 0$

$\frac{\partial f}{\partial n} = \nabla f \cdot \hat{\mathbf{n}} \quad \hat{\mathbf{n}} = (1,0) \quad \Rightarrow \frac{\partial h}{\partial n} = \nabla f \cdot \hat{\mathbf{n}} = 0$

all paths at (1,0) are locally flat [(1,0) is on the rim]

**(c)** At (1,1) in the $\mathbf{n}$ = (1,1) direction

Unit vector in the direction (1,1) is $\hat{\mathbf{n}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\frac{\partial h}{\partial n} = \nabla h \cdot \hat{\mathbf{n}}$

$(x^2+y^2-.25) = f^{\frac{1}{2}} = 1.75$

$f = (x^2+y^2-.25)^2 = 3.0625 \Rightarrow e^{-f} = h = e^{-3.0625} \approx 0.0468$

$$\begin{aligned}
\nabla h &= -0.0468 \times \{4\times 1.75\mathbf{i} + 4\times 1.75\mathbf{j}\} \\
&= -0.327(\mathbf{i}+\mathbf{j}) \\
\nabla h \cdot \hat{\mathbf{n}} &= -0.327(\mathbf{i}+\mathbf{j}) \cdot \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) \\
&= \frac{-2}{\sqrt{2}} \times 0.327 \\
&\approx -0.463
\end{aligned}$$

**(d)** At $(1,1)$ in the $\mathbf{n} = (2,-1)$ direction

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{2^2 + (-1)^2}}(2,-1) = \left(\frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j}\right) \approx 0.894\mathbf{i} - 0.447\mathbf{j}$$

$$\begin{aligned} \frac{\partial h}{\partial n} = \nabla h \cdot \hat{\mathbf{n}} &= -0.327(\mathbf{i} + \mathbf{j}) \times 0.447(2\mathbf{i} + \mathbf{j}) \\ &= -0.327 \times 0.447 \\ &\approx -0.146 \end{aligned}$$