Question

Consider the two circles

$$A_1 = \{z \in \mathbf{C} \mid |z - (2+2i)| = 4\}$$

and

$$A_2 = \{z \in \mathbf{C} \mid |z - 2| = 3\}.$$

Find the points of intersection of these two circles, and determine the angles between the two circles by determining the angle between their tangent lines at one of the points of intersection, call it ξ. (Which point of intersection you use is your choice.)

Consider now the Möbius transformation $m(z) = \dfrac{z}{z+1}$. Determine the equations of the image circles $m(A_1)$ and $m(A_2)$. Determine the angle between $m(A_1)$ and $m(A_2)$ at $m(\xi)$ by determining the angle between their tangent lines.

Confirm that the angle between A_1 and A_2 at ξ is equal to the angle between $m(A_1)$ and $m(A_2)$ at $m(\xi)$.

Answer

Start by determining the intersection points of A_1 and A_2:

A_1 is given by $|z - (2+2i)| = 4$ which expands out to

$$z\bar{z} - (2+2i)\bar{z} - (2-2i)z - 8 = 0$$

A_2 is given by $|z - 2| = 3$ which expands out to

$$z\bar{z} - 2\bar{z} - 2z - 5 = 0$$

$z\bar{z} - (2+2i)\bar{z} - (2-2i)z - 8 = z\bar{z} - 2\bar{z} - 2z - 5$

simplifies to $-2i\bar{z} + 2iz - 3 = 0$ which gives (setting $z = x + iy$) $y = \dfrac{-3}{4}$.

Plugging back into either (both) of the equations for the circles $A_1 A_2$ we get $x = 2 \pm \dfrac{\sqrt{135}}{4}$.

So $A_1 A_2$ intersect at $\underline{\left(2 \pm \dfrac{\sqrt{135}}{4}\right) - \dfrac{3}{4}i}$

Work at $\underline{z_0 = \left(2 \pm \dfrac{\sqrt{135}}{4}\right) - \dfrac{3}{4}i}$

The radius of A_1 passes through z_0 and $2+2i$ and so the slope of the radius is $m_1 = \dfrac{2-(-\frac{3}{4})}{2-(2+\frac{\sqrt{135}}{4})} = \dfrac{\frac{11}{4}}{-\frac{\sqrt{135}}{4}} = \dfrac{-11}{\sqrt{135}}$

The slope of the tangent line is $M_1 = \dfrac{-1}{m_1}$ and so the equation of the tangent line is:

$$y + \frac{3}{4} = \frac{\sqrt{135}}{11}\left(x - \left(2 + \frac{\sqrt{135}}{4}\right)\right)$$

$$\frac{1}{2i}(z-\bar{z}) + \frac{3}{4} = \frac{\sqrt{135}}{11}\left(\frac{1}{2}(z+\bar{z}) - \left(2 + \frac{\sqrt{135}}{4}\right)\right)$$

$$\frac{-i}{2}z + \frac{i}{2}\bar{z} + \frac{3}{4} = \frac{\sqrt{135}}{22}z + \frac{\sqrt{135}}{22}\bar{z} - \frac{\sqrt{135}}{11}\left(2 + \frac{\sqrt{135}}{4}\right)$$

$$0 = \left(\frac{\sqrt{135}}{22} + \frac{i}{2}\right)z + \left(\frac{\sqrt{135}}{22} - \frac{i}{2}\right)\bar{z} - \frac{2\sqrt{135}}{11} - \frac{135}{44} - \frac{3}{4}$$

$$0 = \left(\frac{\sqrt{135}}{22} + \frac{i}{2}\right)z + \left(\frac{\sqrt{135}}{22} - \frac{i}{2}\right)\bar{z} - \frac{(8\sqrt{135}+168)}{44}$$

$$0 = \beta_1 z + \bar{\beta}_1\bar{z} + \gamma_1 = 0$$

Facts to note: the angle that the tangent line to A_1 at z_0 makes with $\mathbf{R}$ is $\theta_1 = \arctan(M_1) \Rightarrow \theta_1 = 0.8128$.

$$\beta_1 = \frac{\sqrt{135}}{22} + \frac{1}{2}$$

$$\gamma_1 = -\frac{(8\sqrt{135}+168)}{44}$$

The radius of A_2 passes through z_0 and 2 and so the slope of the radius is $m_2 = \dfrac{0-(-\frac{3}{4})}{2-(2+\frac{\sqrt{135}}{4})} = \dfrac{\frac{3}{4}}{-\frac{\sqrt{135}}{4}} = \dfrac{-1}{\sqrt{15}}$

The slope of the tangent line is $\underline{M_2 = \frac{-1}{m_2} = \sqrt{15}}$ and so the equation of the tangent line is:

$$y + \frac{3}{4} = \sqrt{15}\left(x - \left(2 + \frac{\sqrt{135}}{4}\right)\right)$$

$$y + \frac{3}{4} = \sqrt{15}x - 2\sqrt{15} - \frac{\sqrt{15}\cdot\sqrt{135}}{4}$$

$$\frac{-i}{2}(z - \bar{z}) + \frac{3}{4} = \sqrt{15}\left(\frac{1}{2}(z + \bar{z})\right) - 2\sqrt{15} - \frac{45}{4}$$

$$\left(\frac{\sqrt{15}}{2} + \frac{i}{2}\right)z + \left(\frac{\sqrt{15}}{2} - \frac{i}{2}\right)\bar{z} - 2\sqrt{15} - \frac{45}{4} - \frac{3}{4} = 0$$

$$\left(\frac{\sqrt{15}}{2} + \frac{i}{2}\right)z + \left(\frac{\sqrt{15}}{2} - \frac{i}{2}\right)\bar{z} - 2\sqrt{15} - 12 = 0$$

$$0 = \beta_2 z + \bar{\beta}_2\bar{z} + \gamma_2 = 0$$

$$\underline{\beta_2 = \frac{\sqrt{15}}{2} + \frac{i}{2}, \quad \gamma_2 = -2\sqrt{15} - 12}$$

The angle that the tangent line to A_2 at z_0 makes with $\mathbf{R}$ is $\theta_2 = \arctan(M_2) \Rightarrow \theta_2 = 1.3181$.

So, the angle between A_2 and A_1 at z_0 is $\underline{\theta_2 - \theta_1 = 0.5053}$.

Equation of m (tangent line to A_k):
$w = m(z) = \frac{z}{z+1}, \quad z = \frac{w}{1-w}$

$$\begin{aligned}
0 &= \beta_k z + \bar{\beta}_k\bar{z} + \gamma_k = 0 \\
&= \beta_k\frac{w}{1-w} + \bar{\beta}_k\frac{\bar{w}}{1-\bar{w}} + \gamma_k \\
0 &= \beta_k w(1-\bar{w}) + \bar{\beta}_k\bar{w}(1-w) + \gamma_k(1-w)(1-\bar{w}) \\
0 &= \beta_k w - \beta_k w\bar{w} + \bar{\beta}_k\bar{w} - \bar{\beta}_k w\bar{w} + \gamma_k - \gamma_k w - \gamma_k\bar{w} + \gamma_k w\bar{w} \\
0 &= (\gamma_k - \beta_k - \bar{\beta}_k)w\bar{w} + (\beta_k - \gamma_k)w + (\bar{\beta}_k - \gamma_k)\bar{w} + \gamma_k
\end{aligned}$$

Center of m(tangent line to A_k at z_0):

$$\frac{\gamma_k - \bar{\beta}_k}{\gamma_k - \beta_k - \bar{\beta}_k} = \frac{\gamma_k - \bar{\beta}_k}{\gamma_k - 2\mathrm{Re}\beta_\mathrm{k}}$$

$\underline{k=1}$ $\gamma_1 = -\dfrac{(8\sqrt{135}+168)}{44}, \quad \beta_1 = \dfrac{\sqrt{135}}{22} + \dfrac{i}{2}$

$$\begin{aligned} \text{center} &= \frac{-(8\sqrt{135}+168) - 2\sqrt{135} + 22i}{-(8\sqrt{135}+168) - 4\sqrt{135}} \\ &= \frac{-10\sqrt{135} - 168 + 22i}{-12\sqrt{135} - 168} \\ &\sim \underline{0.9244 - 0.0716i} \end{aligned}$$

$\underline{k=1}$ $\gamma_2 = -2\sqrt{15} - 12, \quad \beta_2 = \dfrac{\sqrt{15}}{2} + \dfrac{i}{2}$

$$\begin{aligned} \text{center} &= \frac{-2(2\sqrt{15}+12) - (\sqrt{15} - i)}{-2(2\sqrt{15}+12) - 2\sqrt{15}} \\ &= \frac{-5\sqrt{15} - 24 + i}{-6\sqrt{15} - 24} \\ &\sim \underline{0.9180 - 0.0212i} \end{aligned}$$

$$\begin{aligned} m(z_0) = \frac{z_0}{z_0+1} &= \frac{\left(2 + \frac{\sqrt{135}}{4}\right)\left(3 + \frac{\sqrt{135}}{4}\right) + \frac{9}{16} - \frac{3}{4}i}{\left(3 + \frac{\sqrt{135}}{4}\right)^2 + \frac{9}{10}} \\ &\sim 0.8333 - 0.0212i \end{aligned}$$

slope of radius of m (tangent line to A_1): -0.5532
slope of tangent to m (tangent line to A_1): 1.8075
angle of tangent to $\mathbf{R}$: $\phi_1 = 1.0655$

slope of radius of m (tangent line to A_1): ~ 0
slope of tangent to m (tangent line to A_1): $\sim \infty$
angle of tangent to $\mathbf{R}$: $\phi_2 = 1.57 - 8$
and so the angle between $m(A_2)$ and $m(A_1)$ at z_0 is

$$\phi_2 - \phi_1 = 0.5053$$

(as desired)