Question

Determine all the (complex) values of s for which the four points $1+i$, $2i$, $-3-2i$, and s lie on a circle in $\overline{\mathbf{C}}$.

Answer

We use the cross ratio. Specifically the cross ratio is real if and only if the 4 points lie on a circle in $\mathbf{C}$:

$f(1+i, 2i, -3-2i, s)$

$$= \frac{1+i-s}{1+i-2i} \cdot \frac{-3-2i-2i}{-3-2i-s} \text{ write } s = x+iy$$

$$= \frac{1-x+i(1-y)}{1-i} \cdot \frac{-3-4i}{(-3-x)+i(-2-y)}$$

$$= \frac{1-x+i(1-y)}{(-3-x)+i(-2-y)} \cdot \frac{-3-4i}{1-i} \cdot \frac{1+i}{1+i}$$

$$= \frac{(1-x)+i(1-y)}{(-3-x)+i(-2-y)} \cdot \frac{1-7i}{2} \cdot \frac{(-3-x)-i(-2-y)}{(-3-x)-i(-2-y)}$$

$$= \frac{(1-7i)}{2}$$

$$\cdot \frac{[(1-x)(-3-x)+(1-y)(-2-y)+i((1-y)(-3-x)-(1-x)(-2-y))]}{(-3-x)^2+(-2-y)^2}$$

Imaginary part is: (numerator only)

$= -7[(1-x)(-3-x)+(1-y)(-2-y)] + (1-y)(-3-x) + (1-x)(+2+y)$

$= -7(-3+2x+x^2+-2+y+y^2) + (-3+3y-x+xy+2-2+xy-xy)$

$= 21-14x-7x^2+14-7y-7y^2-1+4y-3x$

$= -7x^2-17x-7y^2-3y+34$

$= -7(x^2+\frac{17}{7}x) - 7(y^2+\frac{3}{7}y)+34$

$= -7((x+\frac{17}{14})^2-(\frac{17}{14})^2) - 7((y+\frac{3}{14})^2-(\frac{3}{14})^2)+34$

This equation describes a circle in $\mathbf{C}$; any point on this circle gives a real cross ratio. A number is real if and only if its complex part is 0:

So s gives a real cross ratio if and only if:

$$\underline{\left(x+\frac{17}{14}\right)^2 + \left(y+\frac{3}{14}\right)^2 = \frac{1250}{14^2}. \quad (s = x+iy)}$$