Question
Let $m(z)$ be a loxodromic Möbius transformation that does not fix ∞. The *isometric circle* of m is the circle $\{z \in \mathbf{C} \mid |m'(z)| = 1\}$, where $m'(z)$ is the derivative of $m(z)$ in terms of z. Prove the following statement, or give a counterexample: the isometric circle of a loxodromic Möbius transformation (not fixing ∞) $m(z)$ is always disjoint from the isometric circle of its inverse $m^{-1}(z)$.

[Hint: use the standard form for a loxodromic Möbius transformation, and try calculating a few specific numerical examples.]

Answer
The standard form of a loxodromic fixing $x,\ y$ with multiplier k^2 is
$m(z) = \dfrac{(xk^{-1} - yk)z + xy(k - k^{-1})}{(k - k^{-1})z + xk + yk^{-1}}$
$\left[\text{set } \mathrm{p(z)} = \dfrac{\mathrm{z - x}}{\mathrm{z - y}} \text{ and } \ell(\mathrm{z}) = \mathrm{k^2 z} \text{ and evaluate } \mathrm{p^{-1}}\ell\mathrm{p(z)}\right]$
$\det(m) = (x - y)^2$
$m(z) = \dfrac{\frac{xk^{-1}-yk}{x-y} z + \frac{xy(k-k^{-1})}{x-y}}{\frac{k^{-1}-k}{x-y} z + \frac{xk-yk^{-1}}{x-y}}$ with $\det(m) = 1$.
$m'(z) = \dfrac{1}{\left(\frac{k^{-1}-k}{x-y} z + \frac{xk-yk^{-1}}{x-y}\right)^2}$
$|m'(z)| = 1$:

$$\left|\frac{k^{-1} - k}{x - y} z + \frac{xk - yk^{-1}}{x - y}\right| = 1$$

$$\left|z + \frac{xk - yk^{-1}}{x - y} \cdot \frac{x - y}{k^{-1} - k}\right| = \left|\frac{x - y}{k^{-1} - k}\right|$$

$$\left|z - \left(\frac{yk^{-1} - xk}{k^{-1} - k}\right)\right| = \frac{|x - y|}{|k^{-1} - k|}$$

This is the isometric circle of m.

$$m^{-1}(z) = \frac{\left(\frac{xk - yk^{-1}}{x-y}\right) z - \frac{xy(k-k^{-1})}{x-y}}{\left(\frac{k-k^{-1}}{x-y}\right) z + \frac{xk^{-1} - yk}{x-y}}$$

$$(m^{-1})'(z) = \frac{1}{\left(\left(\frac{k-k^{-1}}{x-y}\right) z + \frac{xk^{-1}-yk}{x-y}\right)^2}$$

$|(m^{-1})'(z)| = 1$:

$$\left| \frac{k-k^{-1}}{x-y} z + \frac{xk^{-1}-yk}{x-y} \right| = 1$$

$$\left| z + \frac{xk^{-1}-yk}{k-k^{-1}} \right| = \left| \frac{x-y}{k-k^{-1}} \right|$$

Isometric circle of $m^{-1}(z)$.
Distance between centers:

$$\begin{aligned} & \left| -\frac{(xk^{-1}-yk)}{k-k^{-1}} - \frac{(yk^{-1}-xk)}{k^{-1}-k} \right| \\ = & \left| \frac{xk^{-1}-yk-yk^{-1}+xk}{k^{-1}-k} \right| \\ = & \left| \frac{(x-y)(k+k^{-1})}{k^{-1}-k} \right| \end{aligned}$$

Sum of radii:

$$\frac{|x-y|}{|k^{-1}-k|} + \frac{|x-y|}{|k^{-1}-k|} = \frac{2|x-y|}{|k^{-1}-k|}$$

Circles are disjoint if and only if distance between centers is greater than sum of radii:

$$\left| \frac{(x-y)(k+k^{-1})}{k^{-1}-k} \right| > \frac{2|x-y|}{|k^{-1}-k|} \quad \text{i.e. } \underline{|k+k^{-1}| > 2}.$$

There are complex numbers k so that $|k| > 1$ and $|k + k^{-1}| > 2$ and so there are loxodromic $m(z)$ for which the isometric circles of m and m^{-1} are not disjoint. (Note that the conditions on m for this are independent of the fixed points.)