Question
Find an integrating factor and hence solve the equation

$$\left(2xy + x^2y + \frac{1}{3}y^3\right) dx + (x^2 + y^2)dy = 0$$

Answer

$$\left(2xy + x^2y + \frac{1}{3}y^3\right) dx + (x^2 + y^2)dy = 0$$

$$P = 2xy + x^2y + \frac{1}{3}y^3 \qquad Q = x^2 + y^2$$

$$\frac{\partial P}{\partial y} = 2x + x^2 + y^2 \qquad \frac{\partial Q}{\partial x} = 2x$$

So $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$ so not exact.
However it fails to be exact by $\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x} = x^2 + y^2 = Q$ so multiply by e^x

$$\left(2xy + x^2y + \frac{1}{3}y^3\right) e^x dx + (x^2 + y^2)e^x dy = 0$$

NOW

$$\tilde{P} = \left(2xy + x^2y + \frac{1}{3}y^3\right) e^x \qquad \tilde{Q} = (x^2 + y^2)e^x$$

$$\frac{\partial \tilde{P}}{\partial y} = (2x + x^2 + y^2)e^x \qquad \frac{\partial \tilde{Q}}{\partial x} = (2x + x^2 + y^2)e^x$$

So $\dfrac{\partial \tilde{P}}{\partial y} = \dfrac{\partial \tilde{Q}}{\partial x}$ and the equation is now exact.
So try and find $F(x, y)$ such that:

$$\frac{\partial F}{\partial x} = \tilde{P} = (2xy + x^2y + \frac{1}{3}y^3)e^x \qquad (1)$$
$$\frac{\partial F}{\partial y} = \tilde{Q} = (x^2 + y^2)e^x \qquad (2)$$

Integrating (2) gives

$$\begin{aligned} F &= (x^2y + \frac{1}{3}y^3)e^x + f(x) \\ \frac{\partial F}{\partial x} &= (2xy + x^2y + \frac{1}{3}y^3)e^x + f'(x) \\ &\Rightarrow f'(x) = 0 \Rightarrow f(x) = c \\ \text{so } F &= (x^2y + \frac{1}{3}y^3)e^x + c \end{aligned}$$

and the solution is $F(x, y) = \text{constant}$.

$$\text{i.e.} \quad \left(x^2y + \frac{1}{3}y^3\right) e^x = K$$