Question
Solve the following non-linear differential equations

(a) $x^2y^2y' - (1+x)(1+y) = 0$

(b) $y' = 2e^{\frac{y}{x}} + \dfrac{y}{x}$

(c) $(y\sec^2 x + \sec x \tan x) + (\tan x + 2y)y' = 0$

(d) $(x^2-1)y' + 2xy^2 = 0, \qquad y(\sqrt{2}) = \dfrac{1}{2}$

Answer

(a) A separable ODE

$$\begin{aligned}
x^2y^2y' - (1+x)(1+y) &= 0 \\
\Rightarrow x^2y^2\frac{dy}{dx} &= (1+x)(1+y) \\
\Rightarrow \int \frac{y^2}{1+y}dy &= \int \frac{1+x}{x^2}dx \\
\text{Now} \quad \int \frac{y^2}{1+y}dy &= \int \left(y - 1 + \frac{1}{y+1}\right) dy \\
&= \frac{1}{2}y^2 - y + \ln(y+1) \\
\text{so} \quad \frac{1}{2}y^2 - y + \ln(y+1) &= \int \left(\frac{1+x}{x^2}\right) dx \\
&= -\frac{1}{x} + \ln x + c \\
\text{i.e.} \quad \frac{1}{2}y^2 - y + \ln(y+1) &= -\frac{1}{x} + \ln x + c
\end{aligned}$$

(b) A homogeneous ODE in x and y

$$\frac{dy}{dx} = 2e^{\frac{y}{x}} + \frac{y}{x} \qquad (1)$$

Let $\dfrac{y}{x} = v$, $y = vx$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

So (1) becomes

$$\begin{aligned} v + x\frac{dv}{dx} &= 2e^v + v \\ \Rightarrow x\frac{dv}{dx} &= 2e^v \\ \Rightarrow \int e^{-v} &= 2\int \frac{dx}{x} \\ -e^{-v} &= 2\ln x + c \\ -e^{-\frac{y}{x}} &= 2\ln x + c \end{aligned}$$

(c) An exact ODE

$$(y\sec^2 x + \sec x \tan x) + (\tan x + 2y)\frac{dy}{dx} = 0$$

$$P = y\sec^2 x + \sec x \tan x \qquad Q = \tan x + 2y$$

$\dfrac{\partial P}{\partial y} = \sec^2 x = \dfrac{\partial Q}{\partial x}$ so the equation is exact.

Now we need to find $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = P = y\sec^2 x + \sec x \tan x \qquad (1)$$

$$\frac{\partial F}{\partial y} = Q = \tan x + 2y \qquad (2)$$

Integrating (2) gives $F = y\tan x + y^2 + f(x)$ (3)

Differentiating (3) w.r.t. x gives $\dfrac{\partial F}{\partial x} = y\sec^2 x + f'(x)$ (4)

So

$$\begin{aligned} f'(x) &= \sec x \tan x \\ \Rightarrow f(x) &= \sec x + c \\ \Rightarrow F &= y\tan x + y^2 + \sec x + c \end{aligned}$$

and so the solution is $F(x, y) =$constant

$$\text{i.e.} \quad y\tan x + y^2 + \sec x + c = K$$

(d)

$$\begin{aligned}(x^2-1)y'+2xy^2 &= 0\\ \Rightarrow (x^2-1)\frac{dy}{dx} &= -2xy^2\\ \Rightarrow -\int\frac{dy}{y^2} &= \int\frac{2x}{x^2-1}dx\\ \Rightarrow \frac{1}{y} &= \ln(x^2-1)+C\end{aligned}$$

when $x=\sqrt{2}, y=\frac{1}{2} \Rightarrow 2=C$ so the answer is

$$y=\ln(x^2-1)+2$$