Question

Solve the initial value problems:

(a) $y' + (\tan x)y = \sin 2x, \qquad y(0) = 1$

(b) $x^2y' + 2xy - x + 1 = 0, \qquad y(1) = 0$

Answer

(a) $y' + \tan x\, y = \sin 2x \qquad (1)$

Integrating factor $e^{\int \tan x\, dx} = e^{\ln(-\cos x)} = \dfrac{1}{\cos x}$

so (1) becomes

$$\begin{aligned} \frac{y'}{\cos x} + \frac{\sin x}{\cos^2 x}y &= \frac{\sin 2x}{\cos x} \\ \Rightarrow \frac{d}{dx}\left(\frac{y}{\cos x}\right) = 2\sin x & \\ \Rightarrow \frac{y}{\cos x} &= \int 2\sin x\, dx \\ &= -2\cos x + c \\ \Rightarrow y &= -2\cos^3 x + c\cos x \end{aligned}$$

Now $y(0) = 1 \Rightarrow 1 = -2 + c$ so $c = 3$ and

$$y = -2\cos^2 x + 3\cos x = \cos(3 - 2\cos x)$$

(b) $x^2y' + 2xy - x + 1 = 0 \quad (1) \Rightarrow y + \dfrac{2}{x}y = \dfrac{x-1}{x^2} \qquad (2)$

Integrating factor $e^{\int \frac{2}{x}\, dx} = e^{2\ln x} = x^2$

so (2) becomes

$$\begin{aligned} y'x^2 + 2xy &= x - 1 \\ \Rightarrow \frac{d}{dx}\left(x^2y\right) &= x - 1 \\ \Rightarrow x^2y &= \int (x-1)\, dx \\ &= \frac{1}{2}x^2 - x + c \\ \Rightarrow y &= \frac{1}{2} - \frac{1}{x} + \frac{c}{x^2} \end{aligned}$$

Now $y(0) = 1 \Rightarrow 0 = \frac{1}{2} - 1 + c$ so $c = \frac{1}{2}$ and

$$y = \frac{1}{2} - \frac{1}{x} + \frac{1}{2x^2}$$