**Question**

Solve the following first order linear differential equations:

**(a)** $xy' + y = \sin x$

**(b)** $(1 - x^2)y' + xy = x$

**(c)** $y' = (y - 1)\cot x$

**(d)** $x^3 y' + (2 - 3x^2)y = x^3$

**Answer**

**(a)** $xy' + y = \sin x \quad (1) \quad \Rightarrow \quad y' + \dfrac{y}{x} = \dfrac{\sin x}{x} \quad (2)$

Integrating Factor $e^{\int \frac{1}{x}dx} = e^{(lnx)} = x$

So (2) becomes

$$\begin{aligned} xy' + y &= \sin x \\ \Rightarrow \frac{d}{x}(xy) &= \sin x \\ \Rightarrow xy &= \int \sin\ dx \\ &= -\cos x + c \\ y &= -\frac{\cos x}{x} + \frac{c}{x} \end{aligned}$$

**(b)** $(1 - x^2)y' + xy = x \quad (1) \quad \Rightarrow \quad y' + \dfrac{x}{1 - x^2}y = \dfrac{x}{1 - x^2} \quad (2)$

Integrating Factor $e^{\int \frac{x}{1-x^2}dx} = e^{(-\frac{1}{2}\ln(1-x^2))} = (1 - x^2)^{-\frac{1}{2}}$

so (2) becomes

$$\begin{aligned} \frac{y'}{(1-x^2)^{\frac{1}{2}}} + \frac{x}{(1-x^2)^{\frac{3}{2}}}y &= \frac{x}{(1-x^2)^{\frac{3}{2}}} \\ \Rightarrow \frac{d}{dx}\left(\frac{y}{(1-x^2)^{\frac{1}{2}}}\right) &= \frac{x}{(1-x^2)^{\frac{3}{2}}} \\ \Rightarrow \frac{y}{(1-x^2)^{\frac{1}{2}}} &= \int \frac{x}{(1-x^2)^{\frac{3}{2}}}dx \\ &= (1-x^2)^{-\frac{1}{2}} + c \\ \Rightarrow y &= 1 + c\sqrt{1-x^2} \end{aligned}$$

**(c)** $y' = (y-1)\cot x \quad (1) \quad \Rightarrow \quad y' - \cot(x)y = -\cot x \quad (2)$

Integrating factor $e^{\int -\cot x\,dx} = e^{-\ln(\sin x)} = (\sin x)^{-1}$

so (2) becomes

$$\begin{aligned}
\frac{y'}{\sin x} + \frac{\cos x}{\sin^2 x}y &= -\frac{\cos x}{\sin^2 x} \\
\Rightarrow \frac{d}{dx}\left(\frac{y}{\sin x}\right) &= -\frac{\cos x}{\sin^2 x} \\
\Rightarrow \frac{y}{\sin x} &= -\int \frac{\cos x}{\sin^2 x}dx \\
&= \frac{1}{\sin x} + c \\
\Rightarrow y &= 1 + c\sin x
\end{aligned}$$

**(d)** $x^3y' + (2-3x^2)y = x^3 \quad (1) \quad \Rightarrow \quad y' + \left(\frac{2}{x^3} - \frac{3}{x}\right)y = 1 \quad (2)$

Integrating factor $e^{\left(\frac{2}{x^3}-\frac{3}{x}\right)dx} = e^{-\frac{1}{x^2}-3\ln x} = e^{-\frac{1}{x}}e^{-3\ln x} = \frac{1}{x^3}e^{-\frac{1}{x^2}}$

so (2) becomes

$$\begin{aligned}
\frac{y'}{x^3}e^{-\frac{1}{x^2}} + \left(\frac{2}{x^6} - \frac{3}{x^4}\right)e^{-\frac{1}{x^2}}y &= \frac{1}{x^3}e^{-\frac{1}{x^2}} \\
\Rightarrow \frac{d}{dx}\left(\frac{y}{x^3}e^{-\frac{1}{x^2}}\right) &= \frac{1}{x^3}e^{-\frac{1}{x^2}} \\
\Rightarrow \frac{y}{x^3}e^{-\frac{1}{x^2}} &= \int \frac{1}{x^3}e^{-\frac{1}{x^2}}\,dx \\
&= \frac{1}{2}e^{-\frac{1}{x^2}} + c \\
\Rightarrow y &= \frac{x^3}{2} + cx^3e^{\frac{1}{x^2}}
\end{aligned}$$