QUESTION

Let C denote the unit circle $z = e^{i\theta} \quad (-\pi < \theta \leq \pi)$. Show that for any real constant a

$$\int_C \frac{e^{az}}{z} dz = 2\pi i$$

and then by writing the integral in terms of θ derive the formula

$$\int_0^\pi e^{a\cos\theta} \cos(a\sin\theta) d\theta = \pi.$$

ANSWER

In (*), $f(z) = e^{az}$, $b = 0$. Thus $\int_C \frac{e^a z}{z} = 2\pi i e^0 = 2\pi i$. As C is the unit circle that can be parameterized as $\{e^{i\theta} | -\pi < \theta \leq \pi\}$ We put $z = e^{i\theta} = \cos\theta + i\sin\theta$ and get $\int_{-\pi}^{\pi} e^{a(\cos\theta + i\sin\theta)} d\theta = 2\pi$. Hence $\int_{-\pi}^{\pi} e^{a\cos\theta} e^{ia\sin\theta} d\theta = 2\pi$. Thus

$$\int_{-\pi}^{\pi} e^{a\cos\theta} \cos(a\sin\theta) d\theta + i \int_{-\pi}^{\pi} e^{a\cos\theta} \sin(a\sin\theta) d\theta = 2\pi$$

Now equate real parts and use the fact that the first integrand is even to get the solution.