QUESTION

Find the value of the integral of $g(z)$ around the circle $|z-i|=2$ where

(a) $g(z)=\frac{1}{z^2+4}$

(b) $g(z)=\frac{1}{(z^2+4)^2}$

ANSWER

Here C is the circle $|z-i|=2$. The singular points of $g(z)$ in both cases are $2i$ and $-2i$, and $2i$ lies within C whilst $-2i$ lies outside C. Thus

(a)

$$\int_C \frac{dz}{z^2+4}=\int_C \frac{dz}{(z+2i)(z-2i)}=\int_C \frac{f(z)}{z-2i},$$

where $f(z)=\frac{1}{z+2i}$. Thus the integral is equal to $2\pi i(f(2i)=\frac{\pi}{2}$

(b) By the same method we find that the integral is equal to $\int_C \frac{h(z)dz}{(z-2i)^2}$ where $h(z)=\frac{1}{(z+2i)^2}$. Thus by the Cauchy integral formula ((*) with $n=1$)the answer is $2\pi i h'(2i)=\frac{\pi}{16}$.