QUESTION

(a) Describe how artificial arcs are used in the network simplex method. Suppose that artificial arcs remain in the tree solution at the end of phase 1 of the two-phase network simplex method. State the conditions under which phase 2 should be performed, and describe what happens to these artificial arcs in phase 2.

(b) Show that the following linear programming problem can be formulated as a minimum cost network flow problem.

$$
\begin{array}{ll}
\text{Minimize} & z = 5x_1 + 8x_2 + 11x_3 + 10x_4 \\
 & +4x_5 + 9x_6 + 6x_7 + 8x_8 + 7x_9 \\
\text{subject to} & x_1, \ldots, x_9 \geq 0 \\
 & x_1 + x_2 = 15 \\
 & x_2 + x_3 + x_4 = 20 \\
 & x_4 + x_5 = 12 \\
 & x_6 + x_7 + x_8 = 27 \\
 & x_8 + x_9 = 14 \\
 & x_3 + x_7 \leq 18.
\end{array}
$$

Starting with a solution in which x_1, x_2, x_4, x_7 and x_8 take positive values, and the constraint $x_3 + x_7 \leq 18$ is satisfied as a strict inequality, use the network simplex method to solve the problem.

ANSWER

(a) Select any node w of the network. for each source node i $(i \neq w)$, if there is no arc from i to w, add an artificial arc with cost $c'_{iw} = 1$. For each non-source node j $(j \neq w)$, if there is no arc from w to j, add an artificial arc with cost $c'_{wj} - 1$. All original arcs (k, l) are assigned a cost $c'_{kl} = 0$. The first phase of the nethod minimizes $z' = \sum c_{kl} x_{kl}$ (where x_{kl} is the flow in arc (k, l)). The initial tree solution is

$$
\begin{aligned}
x_{iw} &= a_i \text{ for source nodes with supply } a_i \\
x_{wj} &= b_j \text{ for intermediate and sink nodes with demand } b_j
\end{aligned}
$$

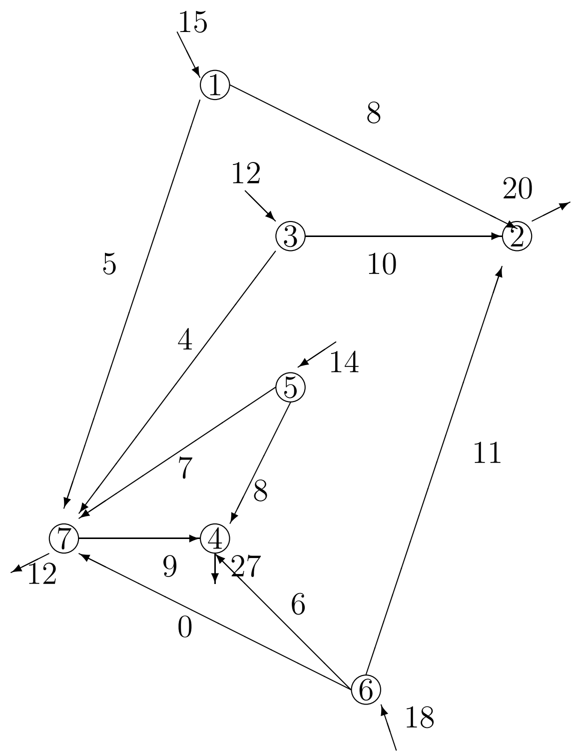
- If, at the end of phase 1, $z' > 0$, the problem is infeasible.
- If $z' = 0$ and the feasible tree solution contains no artificial arc, the second phase of finding the minimum cost flow starts.

- If $z' = 0$ but there is some artificial arc (u, v) in the feasible tree solution with $x_{uv} = 0$, consider the dual variables y_i for this solution. The satisfy $y_i + c'_{ij} \geq y_j$ for all arcs (i, j) with equality holding for arcs of the tree solution. Partition the nodes into two sets S and T, where $S = \{k | y_k \leq y_u\}$ and $T = \{k | y_K > y_u\}$. The problem decomposes into subproblems involving nodes of S, and nodes of T, and arcs between S and T are removed.

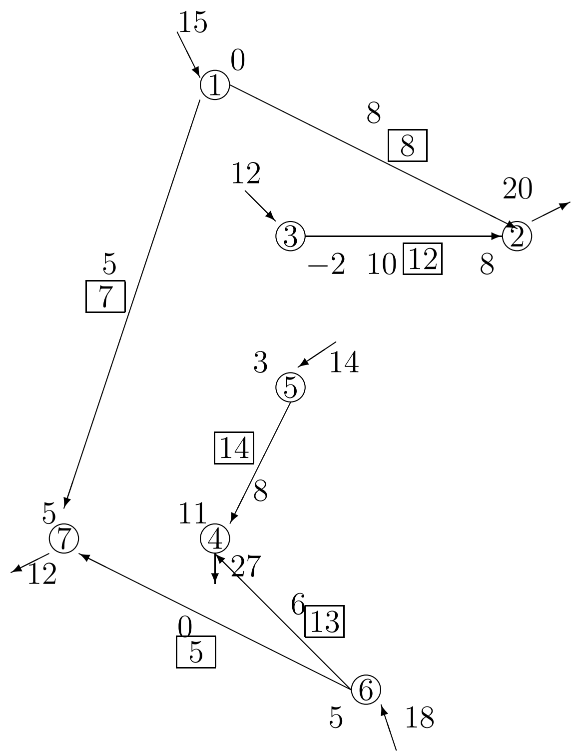
(b) Multiplying some of the constraints by -1 and introducing slack variables, we obtain

$$\begin{aligned} x_1 + x_2 &= 15 \\ -x_2 - x_3 - x_4 &= -20 \\ x_4 + x_5 &= 12 \\ -x_6 - x_7 - x_8 &= -27 \\ x_8 + x_9 &= 14 \\ x_3 + x_7 + s &= 18 \\ -x_1 - x_5 + x_6 - x_9 - s &= -12 \end{aligned}$$

where the last constraint is obtained by summing the others.

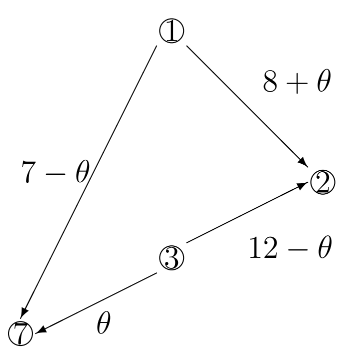


The initial tree solution is



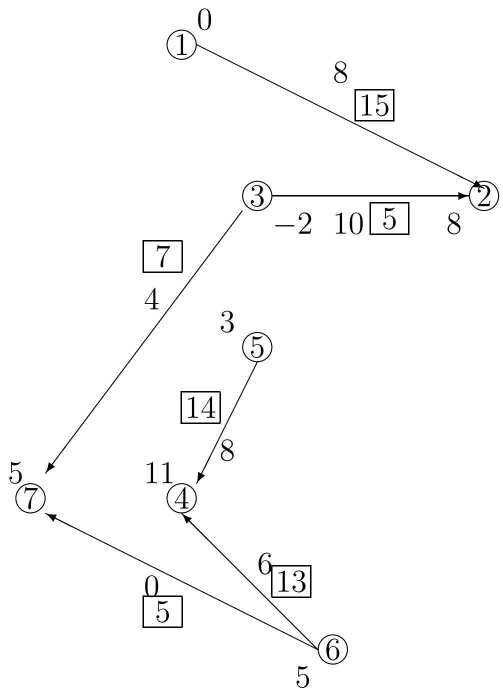
Non basic (i,j)	$y_i + c_{ij} - y_j$
(3,7)	−3
(5,7)	−2
(6,2)	
(7,4)	3

Entering arc is (3,7)



$\theta = 7$

Leaving arc is (1,7).



Non basic (i, j)	$y_i + c_{ij} - y_j$
(1,7)	3
(5,7)	5
(6,2)	5
(7,4)	3

Thus, we have an optimal solution

$$x_1 = 0 \; x_2 = 15 \; x_3 = 0 \; x_4 = 5 \; x_5 = 7 \; x_6 = 0 \; x_7 = 13 \; x_8 = 14 \; x_9 = 0$$

$$z = 8 \times 15 + 10 \times 5 + 4 \times 7 + 6 \times 13 + 8 \times 14 = 388$$