QUESTION

(a) Describe briefly the advantages of the revised simplex method over the original simplex method in which the full tableau is computed at each iteration.

(b) Solve the following linear programming problem using the revised simplex method.

$$\begin{array}{ll} \text{maximize} & z = 9x_1 - 4x_2 + 6x_3 + 8x_4 \\ \text{subject to} & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 \\ & 3x_1 + 3x_2 - 2x_3 + 5x_4 \leq 6 \\ & x_1 - 2x_2 + 2x_3 + 3x_4 \leq 6. \end{array}$$

If the right-hand side of the final constraint changes to $6 + \delta$, find for what range of values of δ the change in the maximum value of z is proportional to δ.

If the objective function coefficient of x_2 changes to $-4 + \delta$, find for what range of values of δ the change in the maximum value of z is proportional to δ.

ANSWER

(a) The amount of computation is significantly less for the revised simplex method when the number of constraints is small compared to the number of variables. The inaccuracies due to rounding errors in the original simplex method are avoided in the revised simplex method if the basis matrix is reinverted at regular periods. The revised simplex method allows special routines for sparse matrix manipulations to be exploited when the original constraint matrix is sparse.

(b)

$$A = \begin{pmatrix} 3 & 3 & -2 & 5 & 1 & 0 \\ 1 & -2 & 2 & 3 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

$$\mathbf{c}^T = \begin{pmatrix} 9 & -4 & 6 & 8 & 0 & 0 \end{pmatrix}$$

Let x_5, x_6 be the two slack variables

x_5	1	0	6
x_6	0	1	6

Iteration 1

$$\mathbf{c}_B^T A_B^{-1} = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$(\mathbf{c}_B^T A_B^{-1} A_N - \mathbf{c}_N^T)\mathbf{x}_N = (0-9)x_1$$

Entering variable is x_1

$$A_B^{-1}\mathbf{a}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \begin{matrix} \text{RHS} & \text{Ratio} \\ 6 & 2 \\ 6 & 6 \end{matrix}$$

Leaving variable is x_5

$$\begin{array}{c|cc|c} \hline x_1 & \frac{1}{3} & 0 & 2 \\ x_5 & -\frac{1}{3} & 1 & 4 \\ \hline \end{array}$$

Iteration 2

$$\mathbf{c}_B^T A_B^{-1} = \begin{pmatrix} 9 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \end{pmatrix}$$

$$(\mathbf{c}_B^T A_B^{-1} A_N - \mathbf{c}_N^T)\mathbf{x}_N = \left(\begin{pmatrix} 3 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} + 4 \right) x_2 + \left(\begin{pmatrix} 3 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \end{pmatrix} - 6 \right) x_3$$

Entering variable is x_3

$$A_B^{-1} a_3 = \begin{pmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{8}{3} \end{pmatrix} \quad \begin{matrix} \text{RHS} & \text{Ratio} \\ 2 & - \\ 4 & \frac{3}{2} \end{matrix}$$

Leaving variable is x_6

$$\begin{array}{c|cc|c} \hline x_1 & \frac{1}{4} & \frac{1}{4} & 3 \\ x_3 & -\frac{1}{8} & \frac{3}{8} & \frac{3}{2} \\ \hline \end{array}$$

Iteration 3

$$\mathbf{c}_B^T A_B^{-1} = \begin{pmatrix} 9 & 6 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{9}{2} \end{pmatrix}$$

$$(\mathbf{c}_B^T A_B^{-1} A_N - \mathbf{c}_N^T)\mathbf{x}_N = \left(\begin{pmatrix} \frac{3}{2} & \frac{9}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} + 4 \right) x_2$$

Entering variable is x_2

$$A_B^{-1} a_2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ -\frac{9}{8} \end{pmatrix} \quad \begin{matrix} \text{RHS} & \text{Ratio} \\ 3 & 12 \\ \frac{3}{2} & - \end{matrix}$$

Leaving variable is x_1

x_2	1	1	12
x_3	1	$\frac{3}{2}$	15

Iteration 4

$$\mathbf{c}_B^T A_B^{-1} = \begin{pmatrix} -4 & \epsilon \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 2 & 5 \end{pmatrix}$$

$$\begin{aligned}
(\mathbf{c}_B^T A_B^{-1} A_N - \mathbf{c}_B^T)\mathbf{x}_N &= \left(\begin{pmatrix} 2 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 9 \right) x_1 \\
&+ \left(\begin{pmatrix} 2 & 5 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} - 8 \right) x_4 \\
&+ \left(\begin{pmatrix} 2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 0 \right) x_5 \\
&+ \left(\begin{pmatrix} 2 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0 \right) x_6 \\
&= 2x_1 + 17x_4 + 2x_5 + 5x_6
\end{aligned}$$

Thus, an optimal solution is

$$x_1 = 0,\ x_2 = 12,\ x_3 = 15,\ x_4 = 0 \quad z = 42$$

The s_2 column of the final tableau is $\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$. Thus, if the last right hand side is $6 + \delta$, the right hand sides in the final tableau are $\begin{pmatrix} 12 + \delta \\ 15 + \frac{3}{2}\delta \\ 42 + 5\delta \end{pmatrix}$ giving $\delta \geq -12$ $\delta \geq -10$

So the required range is $\delta \geq -10$

Use A_B^{-1} to create the full tableau.

Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	0	4	1	0	8	1	1	12
x_3	0	$\frac{9}{2}$	0	1	$\frac{19}{2}$	1	$\frac{3}{2}$	15
	1	2	0	0	17	2	5	42
		$+4\delta$			$+8\delta$	$+\delta$	$+\delta$	$+12\delta$

For the coefficient $(-4+\delta)x_2$, non-negativity in the z row is maintained if δ satisfies $\delta \geq -\frac{17}{8}$ $\delta \geq -2$ $\delta \geq -5$.

Thus the required range is $\delta \geq -\frac{1}{2}$.