QUESTION

(a) Consider the linear programming problem

$$\begin{array}{lll} \text{maximize} & z_P = \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & x_j \geq 0 & j = 1, \ldots, n \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, \ldots, m. \end{array}$$

Write down the dual problem. If the original problem has an optimal solution, what can you say about the optimal value of the dual objective? Prove your result.

(b) Use duality theory to determine whether $x_1 = 0$, $x_2 = 7$, $x_3 = 0$, $x_4 = 2$, is an optimal solution of the linear programming problem

$$\begin{array}{ll} \text{maximize} & z = 11x_1 + 2x_2 + 22x_3 + 24x_4 \\ \text{subject to} & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 \\ & 8x_1 + 4x_2 + 7x_3 - 6x_4 \leq 20 \\ & 5x_1 + 2x_2 + 3x_3 + 5x_4 = 24 \\ & 6x_1 + 3x_2 - 4x_3 - 2x_4 = 17. \end{array}$$

Analyze whether your conclusion remains the same if the objective is changed to minimize z instead of maximize z.

ANSWER

(a) Dual is

$$\begin{array}{ll} \text{minimize} & Z_D = \sum_{i=1}^{m} b_i y_i \\ \text{subject to} & y_i \geq 0 \ i = 1, \ldots, m \\ & \sum_{i=1}^{m} a_{ij} y_i \geq c_j \ j = \ \ldots n \end{array}$$

The duality theorem states that $max\ z_P = min\ z_D$ if the original problem has an optimal solution.

Using matrix notation, the original problem is

$$\begin{array}{ll} \text{maximize} & z_P = \mathbf{c}^T \mathbf{x} \\ \text{subject to} & A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \end{array}$$

where s is a vector of slack variables and the dual is

$$\begin{array}{ll} \text{minimize} & z_D = \mathbf{b}^T \mathbf{y} \\ \text{subject to} & A^T \mathbf{y} \geq \mathbf{c},\ \mathbf{y} \geq \mathbf{0} \end{array}$$

Let the Z_P row for the final simplex table for the original problem be

$$z_P + \overline{\mathbf{c}}^T\mathbf{x} + \overline{\mathbf{d}}^T\mathbf{s} = z*$$

where $\overline{\mathbf{c}} \geq \mathbf{0}$ and $\overline{\mathbf{d}} \geq \mathbf{0}$. Clearly it must be obtained from the original row $z_P\ \mathbf{c}^T\mathbf{x} = 0$ by adding $\mathbf{d}^T(A\mathbf{x} + \mathbf{s} - \mathbf{b})$ to give

$$z_P + (-\mathbf{c}^T + \mathbf{d}^T A)\mathbf{x} + \mathbf{d}^T\mathbf{s} = \mathbf{d}^T\mathbf{b}$$

Thus

$$z* = \mathbf{d}^T\mathbf{b}$$

and

$$\overline{\mathbf{c}}^T = -\mathbf{c}^T + \mathbf{d}^T A$$

Transposing gives

$$\overline{\mathbf{c}} + \mathbf{c}\ A^T\mathbf{d}$$

Consider the dual solution $\mathbf{y} = \mathbf{d} \geq q\mathbf{0}$. It is feasible because

$$A^T\mathbf{y} = A^T\mathbf{d} = \mathbf{c} + \overline{\mathbf{c}} \geq \mathbf{c}$$

since $\overline{\mathbf{c}} \geq \mathbf{0}$.

Also

$$z_P = \mathbf{c}^T\mathbf{x} \leq \mathbf{y}^T A\mathbf{x} \leq \mathbf{y}^T\mathbf{b} = \mathbf{b}^T\mathbf{y} = z_D$$

for any feasible solutions $\mathbf{x}$ and $\mathbf{y}$. Thus if feasible solutions yield $z_P = z_D$, they must be optimal.

(b) For the proposed solution, $z = 62\ s_1 = 4$ where s_1 is the slack variable for the first constraint.

The dual problem is

minimize	$z_D = 20y_1 + 24y_2 + 17y_3$
subject to	$y_1 \geq 0$
	$8y_1 + 5y_2 + 6y_3 - t_1 = 11$
	$4y_1 + 2y_2 + 3y_3 - t_2 = 2$
	$7y_1 + 3y_2 - 4y_3 - t_3 = 22$
	$-6y_1 + 5y_2 - 2y_3 - t_4 = 24$

where $t_1 \geq 0,\ t_2 \geq 0,\ t_3 \geq 0,\ t_4 \geq 0$ are slack variables.

Using complementary slackness, $y_1 = 0,\ t_2 = 0,\ t_4 = 0$

$$\begin{aligned} 2y_2 + 3y_3 &= 2 \\ 5y_2 - 2y_3 &= 24 \end{aligned}$$

$y_2 = 4,\ y_3 = -2.$

This yields $t_1 = -3,\ t_3 = -2$ so the solution is not optimal.

The minimize objective can be represented as

$$\text{minimize } z' = -11x_1 - 2x_2 - 22x_3 - 24x_4$$

so the $z' = -62$ for the proposed solution.

The dual is the same except that the right hand sides of the constraints change sides. Complementary slackness yields the same values, so

$$\begin{aligned} 2y_2 + 3y_3 &= -2 \\ 5y_2 - 2y_3 &= -24 \end{aligned}$$

This yields $y_2 = -4,\ y_3 = 2,\ t_1 = 3,\ t_3 = 2.$

Since $z_D = -62 = z'$, we have an optimal solution.