QUESTION

Let A be the 3×3 symmetric matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Calculate the determinant of the matrix A. Find the eigenvalues of A, and construct the corresponding normalised eigenvectors.

(i) Show that your eigenvalues are mutually orthogonal.

(ii) Write down an orthogonal matrix R such that $R^T AR$ is diagonal with the eigenvalues of A as its diagonal entries. Verify this by calculating AR and $R^T AR$.

ANSWER

Determinant of A:

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 0$$
$$= 2(6-1) - (2) = 8$$

Eigenvalues correspond to solutions of

$$\begin{aligned} \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} &= 0 \\ (2-\lambda)\left[(3-\lambda)(2-\lambda)-1\right] - [2-\lambda] &= 0 \\ (2-\lambda)(\lambda^2 - 5\lambda + 2 - 2) &= 0 \\ (2-\lambda)(\lambda^2 - 5\lambda + 4) &= 0 \\ (2-\lambda)(\lambda-1)(\lambda-4) &= 0 \end{aligned}$$

Therefore the eigenvalues are 1, 2 and 4.

$\lambda = 1$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \left.\begin{aligned} x+y&=0 \\ x+2y+z&=0 \\ y+z&=0 \end{aligned}\right\} \begin{aligned} x &= -y \\ z &= -y \end{aligned}$$

A suitable eigenvector $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ which is normalised to $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$.

$\lambda = 2$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \left.\begin{aligned} y &= 0 \\ x + y + z &= 0 \\ y &= 0 \end{aligned}\right\} \begin{aligned} y &= 0 \\ z &= -x \end{aligned}$$

A suitable eigenvector $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ which is normalised to $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

$\lambda = 4$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \left.\begin{aligned} -2x + y &= 0 \\ x - y + z &= 0 \\ y - 2z &= 0 \end{aligned}\right\} \begin{aligned} y &= 2x \\ y &= 2z \end{aligned}$$

A suitable eigenvector $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ which is m=normalised to $\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$.

(i) Mutually orthogonal eigenvectors $\mathbf{x}_i, \mathbf{x}_j$ satisfy $\mathbf{x}_i.\mathbf{x}_j = 0 = \mathbf{x}_j \mathbf{x}_i$

$$\left.\begin{aligned} \mathbf{x}_1.\mathbf{x}_2 &= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1 + 0 - 1 = 0 \\ \mathbf{x}_1.\mathbf{x}_3 &= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 - 2 + 1 = 0 \\ \mathbf{x}_2.\mathbf{x}_3 &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 + 0 - 1 = 0 \end{aligned}\right\} \text{so eigenvectors are mutually orthogonal.}$$

(ii)

$$R = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \quad R^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$R^T A R \quad = \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$
\begin{aligned}
AR &= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{8}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{2}} & \frac{4}{\sqrt{6}} \end{pmatrix} \\
R^T AR &= \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{8}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{2}} & \frac{4}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}
\end{aligned}
$$