QUESTION

Sketch the region defined by the inequalities $0 \le x \le \pi, 0 \le y \le \pi, 0 \le z \le \pi$.

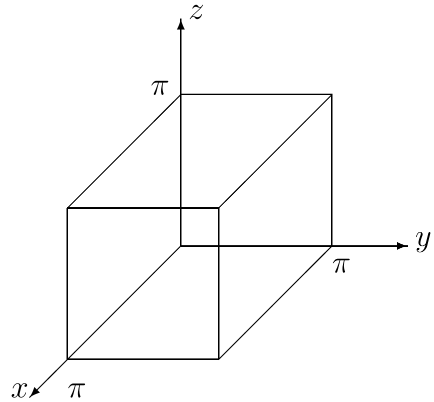
(i) If the region is occupied by a solid S whose density at the point (x, y, z) is $3y^2 z \sin(x)$, calculate the total mass of S using an appropriate triple integral.

(ii) The plane $z = ay$ (where $0 < a \le 1$ is a constant) divides S into two parts: a lower part S_1 lying below the plane, and an upper part S_2 lying above the plane. Sketch S_1 and S_2 when $a = 1$ and when $0 < a < 1$. Find the mass of the lower part S_1 in terms of a.

(iii) Using your answers to (i) and (ii), find the mass of the upper part S_2 in terms of A. Hence show that the mass of S_1 is equal to the mass of S_2 when

$$a = \sqrt{\frac{5}{6}}.$$

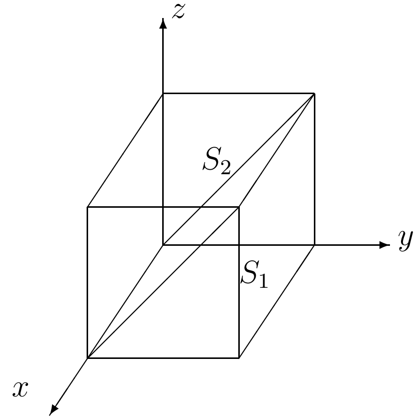
ANSWER



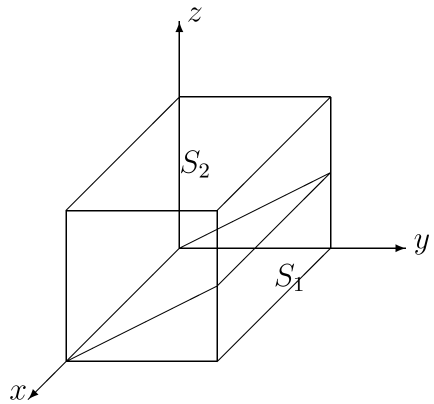
(i) S is a solid cube with side length π. Mass of S:

$$\begin{aligned}
& 3\int_0^\pi z\,dz \int_0^\pi y^2\,dy \int_0^\pi \sin(x)\,dx \\
= & \; 3\left[\frac{1}{2}z^2\right]_0^\pi \left[\frac{1}{3}y^3\right]_0^\pi [-\cos(x)]_0^\pi \\
= & \; 3\left(\frac{1}{2}\pi^2\right)\left(\frac{1}{3}\pi^3\right)(-\cos(\pi)+\cos(0)) \\
= & \; \pi^5
\end{aligned}$$

(ii) when $a = 1,\ z = y$



when $0 < a \leq 1,\ z = ay$



The plane is "hinged" at the x-axis.

$$\begin{aligned}
\text{Mass of } S_1 &= \int_{y=0}^{y=\pi} \int_{z=0}^{z=ay} \int_{x=0}^{x=\pi} 3y^2 z \sin(x)\, dxdydz \\
&= \int_{y=0}^{y=\pi} \int_{z=o}^{z=ay} 6y^2 z\, dzdy \\
&= \int_{y=0}^{y=\pi} \left[3y^2 z^2\right]_{z=0}^{z=ay}\ dy \\
&= \int_{y=0}^{y=\pi} 3a^2 y^4\, dy \\
&= \left[\frac{3}{5} a^2 y^5\right]_0^{\pi}
\end{aligned}$$

$$= \frac{3a^2\pi^5}{5}$$

(iii)

$$\begin{aligned} \text{Mass of } S_2 &= \text{Mass of } s - \text{ mass of } S_1 \\ &= \pi^5 - \frac{3a^2\pi^5}{5} \end{aligned}$$

$$\begin{aligned} \text{mass of } S_1 = \text{mass of } S_2 &\Rightarrow \frac{3a^2\pi^5}{5} = \pi^5 - \frac{3a^2}{5} \\ &\Rightarrow \frac{6a^2\pi^5}{5} = \pi^5 \\ &\Rightarrow a^2 = \frac{5}{6} \\ &\Rightarrow a = \sqrt{\frac{5}{6}} \end{aligned}$$

We take the positive square root since $0 < a \leq 1$.