QUESTION

(a) Sketch the region R defined by the inequalities $x^2 + y^2 \leq 1, x \geq 0$ and $y \geq 0$. Express R in terms of plane polar coordinates. Hence evaluate the following double integral using plane polar coordinates:

$$\iint_R xy\, d(x, y).$$

(HINT: You may assume the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.)

(b) Sketch the region A defined by the inequalities $y \leq 4 - x^2$ and $y \geq (2 - x)^2$. Write the double integral

$$\iint_A 2x\sqrt{y}\, d(x, y)$$

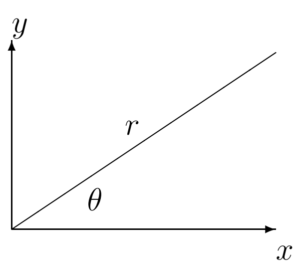
so that the integration with respect to x is performed first. Hence evaluate the integral, and show that it is equal to

$$\frac{2^5}{5}.$$

ANSWER

(a) DIAGRAM

For plane polar coordinates: $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$



R is defined by $\left.\begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq \frac{\pi}{2} \end{array}\right\}$

An area element in plane polar coordinates $rdrd\theta$. So the integral becomes

$$
\begin{aligned}
& \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=1} (r\cos\theta)(r\sin\theta) r dr d\theta \\
= & \int_0^{\frac{\pi}{2}} \int \theta\cos\theta \, d\theta \int_0^1 r^3 \, dr \\
= & \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2\theta \, d\theta \int_0^1 r^3 \, dr \text{ (using } \sin 2\theta = 2\sin\theta\cos\theta) \\
= & \left[-\frac{1}{4}\cos 2\theta\right]_0^{\frac{\pi}{2}} \left[\frac{1}{4}r^4\right]_0^1 \\
= & \left(-\frac{1}{4}\cos\pi + \frac{1}{4}\cos 0\right)\left(\frac{1}{4}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{8}
\end{aligned}
$$

(b) DIAGRAM

To perform integration with respect to x first the limits become

$$
\begin{aligned}
2-\sqrt{y} \leq & \; x \; \leq \sqrt{4-y} \\
0 \leq & \; y \; \leq 4
\end{aligned}
$$

So the integral is

$$
\begin{aligned}
& \int_{y=0}^{y=4} \int_{x=2-\sqrt{y}}^{x=\sqrt{4-y}} 2xy^{\frac{1}{2}} \, dxdy \\
= & \int_{y=0}^{y=4} \left[x^2 y^{\frac{1}{2}}\right]_{x=2-\sqrt{y}}^{x=\sqrt{4-y}} dy \\
= & \int_0^4 (4-y)y^{\frac{1}{2}} - (2-y^{\frac{1}{2}})^2 y^{\frac{1}{2}} \, dy \\
= & \int_0^4 4y^{\frac{1}{2}} - y^{\frac{3}{2}} - y^{\frac{1}{2}}(4+y-4y^{\frac{1}{2}}) \, dy \\
= & \int_0^4 4y - 2y^{\frac{3}{2}} \, dy \\
= & \left[2y^2 - \frac{4}{5}y^{\frac{5}{2}}\right]_0^4 \\
= & \; 2.2^4 - \frac{4}{5}(2^2)^{\frac{5}{3}} \\
= & \; 2^5 - \frac{4}{5}.2^5 = 2^5\left(1-\frac{4}{5}\right) = \frac{2^5}{5}
\end{aligned}
$$