QUESTION

(a) Find the vector equation of the plane Π_1 which passes through the three points $L = (1,3,4)$, $M = (2,-3,0)$ and $N = (1,0,1)$. What is the equation of the plane in terms of x, y, z coordinates?

A second plane Π_2 is parallel to Π_1 and passes through the point $Q = (3,1,1)$. Find the equation of Π_2 in terms of x, y, z coordinates.

(b) (You should state clearly any properties of the cross product that you use.) Let $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ be vectors. Calculate $\mathbf{a} \times \mathbf{b}$ and hence write down $\mathbf{b} \times \mathbf{a}$. Find the relation that must hold between x_1, x_2 and x_3 if the vector $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ is to be written in the form

$$\mathbf{x} = s\mathbf{a} + t\mathbf{b}$$

where s and t are scalars.

If $\mathbf{c} = 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$, calculate $\mathbf{c} \times \mathbf{a}$ and $\mathbf{c} \times \mathbf{b}$. Show that $\mathbf{c}$ can be expressed in the form $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ and find s and t in this case.

ANSWER

(a) Vector equation of plane: $\mathbf{n}(\mathbf{r} - \mathbf{r}_0) = 0$ where

$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is the position vector of a general point on the plane;

$\mathbf{r}_0$ is the position vector of a known point on the plane;

$\mathbf{n}$ is a normal vector to the plane.

The point $L = (1,3,4)$ lies on the plane, so let $\mathbf{r}_0 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$. two vectors in the plane are $\vec{LM}$ and $\vec{LN}$.

$$\vec{LM} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ -4 \end{pmatrix}, \ \vec{LN} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix}$$

A vector normal to the plane is $\vec{LM} \times \vec{LN}$ when

$$\vec{LM} \times \vec{LN} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -6 & -4 \\ 0 & -3 & -3 \end{vmatrix} = \begin{pmatrix} 6 \\ 3 \\ -3 \end{pmatrix}$$

so let $\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

Vector equation of plane Π_1 is therefore $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right] = 0$

Standard equation of $\Pi_1 :\ 2(x-1)+(y-3)-(z-4) = 0$ or $2x+y-z = 2+3-4 = 1$

Plane Π_2 is parallel to Π_1 and so has the equation $2x + y - z = d$ for some constant d.

Since Π_2 contains the point $Q = (3, 1, 1)$ we have $2(3) + (1) - (1) = d \Rightarrow d = 6$ so the standard equation of Π_2 is $2x + y - z = 6$.

(b)

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 2 & 27 & -5 \end{vmatrix} = \begin{pmatrix} 11 \\ 9 \\ 8 \end{pmatrix}, \ \mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) = \begin{pmatrix} -11 \\ -9 \\ -8 \end{pmatrix}$$

Note that $\mathbf{a}.(\mathbf{a} \times \mathbf{b}) = 0 = \mathbf{b}.(\mathbf{a} \times \mathbf{b})$

Take dot product of $\mathbf{x} = s\mathbf{a}+t\mathbf{b}$ with vector $(\mathbf{a}\times\mathbf{b}) = 0$ so $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 11 \\ 9 \\ 8 \end{pmatrix} = 0$ and relation is $11x_1 + 9x_2 + 8x_3 = 0$

$$\mathbf{c} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -5 & -4 \\ 1 & -3 & 2 \end{vmatrix} = \begin{pmatrix} -22 \\ -18 \\ -16 \end{pmatrix}, \ \mathbf{c} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -5 & -4 \\ 2 & 2 & -5 \end{vmatrix} = \begin{pmatrix} 33 \\ 27 \\ 24 \end{pmatrix}$$

$\mathbf{c} = \begin{pmatrix} 7 \\ -5 \\ -4 \end{pmatrix}$ so if $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ then we must have the relation

$$11(7) + 9(-5) + 8) - 4) = 77 - 45 - 32 = 0.$$

so $\mathbf{c}$ can be expanded in the required form.

Note $\mathbf{a} \times \mathbf{a} = 0 = \mathbf{b} \times \mathbf{b}$.

To fins s: take cross product of $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ with $\mathbf{b}$

$$\mathbf{c} \times \mathbf{b} = s\mathbf{a} \times \mathbf{b} + t\mathbf{b} \times \mathbf{b} = s\mathbf{a} \times \mathbf{b}$$

then $\begin{pmatrix} 33 \\ 27 \\ 24 \end{pmatrix} = s \begin{pmatrix} 11 \\ 9 \\ 8 \end{pmatrix}$ and so $s = 3$.

To find t: Take the cross product of $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ with $\mathbf{a}$

$$\mathbf{c} \times \mathbf{a} = s\mathbf{a} \times \mathbf{a} + t\mathbf{b} \times \mathbf{a} = t\mathbf{b} \times \mathbf{a}$$

then $\begin{pmatrix} -22 \\ -18 \\ -16 \end{pmatrix} = t \begin{pmatrix} -11 \\ -9 \\ -8 \end{pmatrix}$ and $t = 2$.

Therefore, $\mathbf{c} = 3\mathbf{a} + 2\mathbf{b}$.