QUESTION

Solve the following system of linear differential equations, subject to the initial conditions $\frac{dx}{dt} = \frac{dz}{dt} = 8$ and $\frac{dy}{dt} = 52$ when $t = 0$:

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ -1 & -6 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

ANSWER

Solution to $\mathbf{x}' = A\mathbf{x}$ where A is an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and the eigenvectors $\mathbf{x}_1, \ldots \mathbf{x}_n$ (each with an arbitrary constant) is

$$\mathbf{x} = \mathbf{x}_1 e^{\lambda_1 t} + \mathbf{x}_2 e^{\lambda_2 t} + \ldots + \mathbf{x}_n e^{\lambda_n t}.$$

Eigenvalues of A:

$$\begin{aligned} \begin{vmatrix} 3-\lambda & 1 & 0 \\ -1 & -6-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} &= 0 \\ (3-\lambda)\left[(2-\lambda)(-6-\lambda) - 1\right] - \left[-(2-\lambda) - 1\right] + 0 &= 0 \\ (3-\lambda)[\lambda^2 + 4\lambda - 13] + (3-\lambda) &= 0 \\ (3-\lambda)(\lambda^2 + 4\lambda - 12) &= 0 \\ (3-\lambda)(\lambda+6)(\lambda-2) &= 0 \end{aligned}$$

Eigenvalues are $\lambda = 2, 3, -6$.

$\lambda = 2$

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & -8 & -1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \left.\begin{aligned} x + y &= 0 \\ -x - 8y - z &= 0 \\ -x - y &= 0 \end{aligned}\right\} \begin{aligned} y &= -x \\ z &= 7x \end{aligned}$$

Let $x = \alpha, y = -\alpha, z = 7\alpha$, a suitable eigenvector is $\mathbf{x}_1 = \begin{pmatrix} \alpha \\ -\alpha \\ 7\alpha \end{pmatrix}$.

$\lambda = 3$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & -9 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \left.\begin{aligned} y &= 0 \\ -x - 9y - z &= 0 \\ -x - y &= 0 \end{aligned}\right\} \begin{aligned} y &= 0 \\ z &= -x \end{aligned}$$

Let $x = \beta, y = 0, z = -\beta$, a suitable eigenvector is $\mathbf{x}_2 = \begin{pmatrix} \beta \\ 0 \\ -\beta \end{pmatrix}$.

$\lambda = -6$

$$\begin{pmatrix} 9 & 1 & 0 \\ -1 & 0 & -1 \\ -1 & -1 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \left.\begin{array}{r} 9x + y = 0 \\ -x - z = 0 \\ -x - y + 8z = 0 \end{array}\right\} \begin{array}{c} y = -9x \\ z = -x \end{array}$$

Let $x = \gamma, y = -9\gamma, z = -\gamma$, a suitable eigenvector is $\mathbf{x}_1 = \begin{pmatrix} \gamma \\ -9\gamma \\ -\gamma \end{pmatrix}$.

Hence the general solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} e^{2t} + \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{3t} + \gamma \begin{pmatrix} 1 \\ -9 \\ -1 \end{pmatrix} e^{-6t}$$

$$\begin{array}{r|r} x = \alpha e^{2t} + \beta e^{3t} + \gamma e^{-6t} & \frac{dx}{dt} = 2\alpha e^{2t} + 3\beta e^{3t} - 6\alpha e^{-6t} \\ y = -\alpha e^{2t} - 9\gamma e^{-6t} & \frac{dy}{dt} = -2\alpha e^{2t} + 54\gamma e^{-6t} \\ z = 6\alpha e^{2t} - \beta e^{3t} - \gamma e^{-6t} & \frac{dz}{dt} = 14\alpha e^{2t} - 3\beta e^{3t} + 6\gamma e^{-6t} \end{array}$$

When $t = 0$, $\frac{dx}{dt} = \frac{dz}{dt} = 8$ and $\frac{dy}{dt} = 52$:

$$\begin{aligned} 8 &= 2\alpha + 3\beta - 6\alpha && (1) \\ 52 &= -2\alpha + 54\gamma && (2) \\ 8 &= 14\alpha - 3\beta + 6\gamma && (3) \end{aligned}$$

(1)+(3)$\Rightarrow 16 = 16\alpha \Rightarrow \alpha = 1$
substitute $\alpha = 1$ into (2) $:52 - 2 + 54\gamma \Rightarrow \alpha = 1$
substitute $\alpha = \gamma = 1$ into (1): $\beta = \frac{1}{3}(8 + 6 - 2) = 4$
Particular solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} e^{2t} + 4 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{3t} + \begin{pmatrix} 1 \\ -9 \\ -1 \end{pmatrix} e^{-6t}.$$