Question
Consider the two dimensional Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with Cauchy boundary conditions:

$$u(x,0) = f(x),\ u_y(x,0) = g(x).$$

Recognising Laplace's equation as a constant coefficient second order linear PDE, show that the general solution is given by

$$u(x,y) = F(x+iy) + G(x-iy)$$

What can you deduce about the characteristaics of this equation? Solve the equation for the given boundary conditions and show that

$$u(x,y) = \frac{1}{2}[f(x+iy) + f(x-iy)] - \frac{i}{2}\int_{x-iy}^{x+iy} g(s)ds.$$

Consider now the specific problems

(i) $f(x) = \dfrac{1}{1+x^2},\ g(x) = 0$

(ii) $f(x) = exp(-x^2),\ g(x) = 0$

Where are the singularities of the boundary conditions? Where are the singularities of the solution $u(x,y)$?
This question demonstrates that although it is possible to construct a solution of Laplace's equation with Cauchy boundary conditions, such solutions often pose problems. Real singularities can appear in the solution that depend on the complex singularities of the boundary conditions or unbounded behaviour can occur in one direction even thought the b.c. was well defined. In fact the only solution that is analytic everywhere is $u = const$. Cauchy boundary conditions are therefore often inappropriate for Laplace's equation with Direchlet or Neumann conditions giving unique bounded solutions in their respective domains of interest.

Answer

Use above method (Q2) or just substitute

$$u = F(x+iy) + G(x-iy) \quad (0)$$

$$\begin{array}{lclcl} u_x &=& F' + G' & ; & u_y = iF' - iG' \\ u_{xx} &=& F'' + G'' & ; & u_{yy} = -F'' - G'' \end{array}$$

$$u_{xx} + u_{yy} = F'' + G'' - F'' - G'' = 0$$

Characteristics are complex.

BC: $u(x,0) = f(x) \Rightarrow F(x) + G(x) = f(x) \quad (1)$

BC: $u(x,0) = g(x) \Rightarrow iF'(x) - iG'(x) = g(x) \quad (2)$

(2) gives $F(x) - G(x) = \frac{1}{i}\int_0^x g(s)\,ds$

Thus

$$\begin{cases} F(x) + G(x) &= f(x) \\ F(x) - G(x) &= -i\int_0^x g(s)ds \end{cases}$$

$\Rightarrow$

$$\left.\begin{array}{lcl} F(x) &=& \frac{1}{2}f(x) - \frac{i}{2}\int_0^x g(s)\,ds \\ G(x) &=& \frac{1}{2}f(x) + \frac{i}{2}\int_0^x g(s)\,ds \end{array}\right\} \quad (3)$$

so inserting (3) in (0)

$$\underline{u = \frac{1}{2}[f(x+iy) + f(x-iy)] - \frac{i}{2}\int_{x-iy}^{x+iy} g(s)\,ds \quad (4)}$$

as required.

(i) singularities at $x \pm i$ (i.e., nowhere real)

(ii) singularities at $x = \pm - \infty$ (i.e., nowhere finite)

substitute (i) into (4); no contribution from integral

$$u = \frac{1}{2}\left[\frac{1}{1+(x+iy)^2} + \frac{1}{1+(x-iy)^2}\right]$$

This is a real and finite function except where $x = 0,\ y = \pm$, where it blows up.

Complex boundary condition singularities here

$\Rightarrow$ real solution singularities.

Substitute (ii) into (4); no contribution from integral,

$$\begin{aligned} u &= \frac{1}{2}[e^{-(x+iy)^2} + e^{-(x-iy)^2}] \\ &= e^{-x^2+y^2}\cos 2xy \end{aligned}$$

This function increases unboundedly for $y > 0$. Thus singularity of boundary condition at infinity leads to bad solution.