**Question**

Confirm the solution of Q6 by showing that it is not possible to calculate the second order derivatives in the Taylor expansion of $u(x,y)$ at the point corresponding to $\alpha$. Then calculate the second order Taylor expansion of the solution about the point $(x,y)=(1,0)$.

**Answer**

$2u_{xx}-5u_{xy}+2u_{yy}+u_x-3u=0$

$$\left.\begin{array}{lrcl} (1) & u(\cos\theta,\sin\theta) & = & \theta \\ (2) & u_x(\cos\theta,\sin\theta) & = & 0 \end{array}\right\} C:(x,y)=(X(\theta),Y(\theta))=(\cos\theta,\sin\theta)$$

Seek $u$ (from (1)), $u_x$ (from (2)), $u_y,\ u_{xx},\ u_{xy},\ u_{yy}$ etc. on curve$C$.
Then use Taylor expansion to find $u(x,y)$ <u>off</u> $C$ as in notes.
First:

$$\begin{array}{lcl} \dot{X}(\theta) & = & -\sin\theta \quad , \quad \dot{Y}(\theta) = \cos\theta \\ \ddot{X}(\theta) & = & -\cos\theta \quad , \quad \ddot{Y}(\theta) = -\sin\theta \end{array}$$

Differentiate boundary conditions

$\dfrac{d(1)}{d\theta}:=$

$u_x|_C\,\dot{X}+u_y|_C\,\dot{Y}=1$

$\Rightarrow \underbrace{0}_{\text{from (2)}}\times(-\sin\theta)+u_y|_C\cos\theta=1$

$$\Rightarrow \underline{u_y|_C=\sec\theta} \quad (3)$$

Thus we now know $u,\ u_x,\ u_y$ on $C$. To find $u_{xx},\ u_{xy},\ u_{yy}$ on $C$ we need 3 equations, the PDE itself is the first:

(4)$2\,u_{xx}|_C-u_{xy}|_C+2\,u_{yy}|_C+u_x|_C-3\,u|_C=0$

$\Rightarrow 2\,u_{xx}|_C-u_{xy}|_C+2\,u_{yy}|_C+0-3\theta=0$

or $\underline{2\,u_{xx}|_C-u_{xy}|_C+2\,u_{yy}|_C=3\theta}$ on $C$ (5)

Second equation comes from:

$\dfrac{d(2)}{d\theta}:=$

$u_{xx}|_C\,\dot{X}+u_{xy}|_C\,\dot{Y}=0$

$\Rightarrow \underline{-u_{xx}|_C\sin\theta+u_{xy}|_C\cos\theta=0}$ (6)

Third equation comes from
$\dfrac{d^2(1)}{d\theta^2} :=$

$$\begin{aligned} & u_{xx}\big|_C \dot{X}^2 + u_{xy}\big|_C \dot{X}\dot{Y} + u_x\big|_C \ddot{X} \\ + \quad & u_y\big|_C \dot{Y}\dot{X} + u_{yy}\big|_C \dot{Y}^2 + u_y\big|_C \ddot{Y} = 0 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} & u_{xx}\big|_C \sin^2\theta - 2\sin\theta\cos\theta \; u_{xy}\big|_C + 0 \times (-\cos\theta) \\ + \quad & u_{yy}\big|_C \cos^2\theta + \sec\theta \times (-\sin\theta) = 0 \end{aligned}$$

$\Rightarrow \underline{\sin^2\theta \; u_{xx}\big|_C - 2\sin\theta\cos\theta \; u_{xy}\big|_C + \cos^2\theta \; u_{yy}\big|_C = \tan\theta} \quad (7)$
Thus we have from (5), (6) and 97)

$$\begin{matrix} (7) & \rightarrow \\ (6) & \rightarrow \\ (5) & \rightarrow \end{matrix} \underbrace{\begin{pmatrix} \sin^2\theta & -2\sin\theta\cos\theta & \cos^2\theta \\ -\sin\theta & \cos\theta & 0 \\ 2 & -5 & 2 \end{pmatrix}}_{\triangle = det()} \begin{pmatrix} u_{xx}\big|_C \\ u_{xy}\big|_C \\ u_{yy}\big|_C \end{pmatrix} = \begin{pmatrix} \tan\theta \\ 0 \\ 3\theta \end{pmatrix}$$

To solve this we need $\triangle \neq 0$.

$$\begin{aligned} & \triangle \\ = \quad & \sin^2\theta \begin{vmatrix} \cos\theta & 0 \\ -5 & 2 \end{vmatrix} \\ & +2\sin\theta\cos\theta \begin{vmatrix} -\sin\theta & 0 \\ 2 & 2 \end{vmatrix} + \cos^2\theta \begin{vmatrix} -\sin\theta & \cos\theta \\ 2 & -5 \end{vmatrix} \\ = \quad & \cos\theta(5\sin\theta\cos\theta - 2) \text{ after algebra} \end{aligned}$$

Thus $\triangle = 0$ when $\cos\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2}, \; -\dfrac{\pi}{2}$, etc.
<u>or</u> $5\sin\theta\cos\theta = 2$.
Now remember that from Q6 $\alpha = \tan^{-1}\dfrac{1}{2}$
PICTURE

$\Rightarrow \sin\alpha = \dfrac{1}{\sqrt{5}}, \; \cos\alpha = \dfrac{2}{\sqrt{5}}$

$$\Rightarrow 5\sin\alpha\cos\alpha = 5\times\frac{1}{\sqrt{5}}\times\frac{2}{\sqrt{5}} = 2(!)$$

Thus it is not possible to find

$$u_{xx}|_C\,,\ u_{xy}|_C\,,\ u_{yy}|_C$$

when $\theta = \alpha = \tan^{-1}\frac{1}{2}$ as found in Q6.

OK, now assume that $0 \le \theta < \alpha$

Then solutions of (8) are given by Cramer's rule (see lecture notes)

$$\left\{\begin{array}{lcl}
u_{xx}|_C &=& \frac{1}{\triangle}\begin{vmatrix} \tan\theta & -2\sin\theta\cos\theta & \cos^2\theta \\ 0 & \cos\theta & 0 \\ 3\theta & -5 & 2 \end{vmatrix} \\
u_{xy}|_C &=& \frac{1}{\triangle}\begin{vmatrix} \sin^2\theta & \tan\theta & \cos^2\theta \\ -\sin\theta & 0 & 0 \\ 2 & 3\theta & 2 \end{vmatrix} \\
u_{yy}|_C &=& \frac{1}{\triangle}\begin{vmatrix} \sin^3\theta & -2\sin\theta\cos\theta & \tan\theta \\ -\sin\theta & \cos\theta & 0 \\ 2 & -5 & 3\theta \end{vmatrix}
\end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{lcl}
u_{xx}|_C &=& \frac{1}{\triangle}\left\{\tan\theta\begin{vmatrix} \cos\theta & 0 \\ -5 & 2 \end{vmatrix} + 2\sin\theta\cos\theta\begin{vmatrix} 0 & 0 \\ 3\theta & 2 \end{vmatrix}\right. \\
&& \left. +\cos^2\theta\begin{vmatrix} 0 & \cos\theta \\ 3\theta & -5 \end{vmatrix}\right\} \\
u_{xy}|_C &=& \frac{1}{\triangle}\left\{\sin^2\theta\begin{vmatrix} 0 & 0 \\ 3\theta & 2 \end{vmatrix} - 2\tan\theta\begin{vmatrix} -\sin\theta & 0 \\ 2 & 2 \end{vmatrix}\right. \\
&& \left. +\cos^2\theta\begin{vmatrix} -\sin\theta & 0 \\ 2 & 3\theta \end{vmatrix}\right\} \\
u_{yy}|_C &=& \frac{1}{\triangle}\left\{\sin^2\theta\begin{vmatrix} \cos\theta & 0 \\ -5 & 3\theta \end{vmatrix} + 2\sin\theta\cos\theta\begin{vmatrix} -\sin\theta & 0 \\ 2 & 3\theta \end{vmatrix}\right. \\
&& \left. +\tan\theta\begin{vmatrix} -\sin\theta & \cos\theta \\ 2 & -5 \end{vmatrix}\right\}
\end{array}\right.$$

$$\left\{\begin{array}{lcl}
u_{xx}|_C &=& \frac{1}{\triangle}\{2\sin\theta - 3\theta\cos^3\theta\} \\
u_{xy}|_C &=& \frac{\tan\theta}{\triangle}\{2\sin\theta - 3\theta\cos^3\theta\} \\
u_{yy}|_C &=& \frac{\tan\theta}{\triangle}\{5\sin\theta - 2\cos\theta - 3\theta\sin\theta\cos^2\theta\}
\end{array}\right. \qquad (9)$$

Hence when $\theta = 0 \Longleftrightarrow (x, y) = (1, 0)$

we have $\triangle = \cos 0(5 \times 0 \times 1 - 2) = -2$
So
$(1) \Rightarrow u(1,0) = 0$
$(2) \Rightarrow u_x(1,0) = 0$
$(3) \Rightarrow u_y(1,0) = \sec 0 = 1$

$$(9) \Rightarrow \begin{cases} u_{xx}(1,0) &= -\frac{1}{2}(0-0) &= 0 \\ u_{xy}(1,0) &= \frac{0}{-2}(0-0) &= 0 \\ u_{yy}(1,0) &= \frac{0}{-2}(0-2-0) &= 0 \end{cases}$$

Thus, to second order

$$\begin{aligned} u(x,y) &= u(1,0) + (x-1)\left.\frac{\partial u}{\partial x}\right|_{(1,0)} \\ &\quad + y\left.\frac{\partial u}{\partial x}\right|_{(1,0)} + \frac{(x-1)^2}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_{(1,0)} \\ &\quad + (x-1)y\left.\frac{\partial^2 u}{\partial x \partial y}\right|_{(1,0)} + y^2\left.\frac{\partial^2 u}{\partial y^2}\right|_{(1,0)} + \cdots \\ &= 0 + (x-1) \times 0 + y \times 1 + 0 + 0 + 0 + \cdots \\ &= \underline{y + \text{3rd order terms!}} \end{aligned}$$