Question

Consider the equation

$$u_{xx} + 8u_{xy} + 7u_{yy} = 0$$

(i) Classify and calculate the characteristics of this equation.

(ii) Let the boundary data

$$u(\tau, \tau) = f(\tau), \; u_x(\tau, \tau) = g(\tau)$$

be given on the curve $C : (x, y) = (X(\tau), Y(\tau))$ where $X(\tau) = Y(\tau) = \tau$ where f and g are sufficiently differentiable functions.

Write down the general form of the Taylor expansion of the solution $u(x, y)$ about a general given point (x_0, y_0) to second order in the derivatives wrt x and y.

By differentiating the boundary data and using the equation itself, show that it is not possible to calculate the second derivatives anywhere, and hence it is not possible to solve the equation with this boundary data. Comment on this in relation to the answers of part (i).

(iii) If now the boundary data is

$$u(X(\tau), Y(\tau)) = f(\tau), \; u_y(X(\tau), Y(\tau)) = g(\tau)$$

and is given on the curve $C : (x, y) = (X(\tau), Y(\tau)) = (\tau, 0)$, calculate the expansion of $u(x, y)$ about $(0, 0)$ up to second order.

Answer

(i) Second order, linear, homogeneous, constant coefficients, PDE

$a = 1, \; b = 4, \; c = 7$

$\Rightarrow b^2 - ac = 16 - 7 = 9 > 0 \Rightarrow$ hyperbolic everywhere

Characteristics given by

$$\frac{dy}{dx} = \frac{4 \pm \sqrt{16 - 7}}{1} = \frac{4 \pm 3}{1} = 7 \text{ or } 1$$

$$\Rightarrow uny = 7x + c \text{ or y = x + d}$$

where $c,\ d$ are arbitrary constants

(Note that general solution from bookwork is

$u(x,y) = P(y-7x) + Q(y-x)$ for $P,\ Q$ arbitrary functions)

(ii)

$$
\begin{aligned}
u(x,y) &= u(x_0,y_0) + (x-x_0)\frac{\partial u}{\partial x}\bigg|_{(x_0,y_0)} + (y-y_0)\frac{\partial u}{\partial y}\bigg|_{(x_0,y_0)} \\
&+ \frac{1}{2}(x-x_0)^2\frac{\partial^2 u}{\partial x^2}\bigg|_{(x_0,y_0)} + (x-x_0)(y-y_0)\frac{\partial^2 u}{\partial x \partial y}\bigg|_{(x_0,y_0)} \\
&+ \frac{1}{2}(y-y_0)^2\frac{\partial^2 u}{\partial y^2}\bigg|_{(x_0,y_0)} + \cdots
\end{aligned}
$$

Taylor expansion to 2nd order:

From lectures we use the following

$$u(x,y) = u(X(\tau) = Y(\tau)) \text{ on C}$$

But when $X(\tau) = Y(\tau) = \tau$

$$
\begin{aligned}
u(X,Y) &= f(\tau) \quad (1) \\
u_x(X,Y) &= g(\tau) \quad (2)
\end{aligned}
$$

Differentiate both wrt τ:

$$
\begin{aligned}
\frac{d(1)}{d\tau} &:= u_x\dot{X} + u_y\dot{Y} = \dot{f} \quad (3) \\
\frac{d(2)}{d\tau} &:= u_{xx}\dot{X} + u_{xy}\dot{Y} = \dot{g} \quad (4)
\end{aligned}
$$

$(\dot{} \equiv \frac{d}{d\tau})$

Therefore on C we have $\dot{X} = 1,\ \dot{Y} = 1$ and so

$(3) : \underline{u_x|_C + u_y|_C = \dot{f}}$

$(4) : \underline{u_{xx}|_C + u_{xy}|_C = \dot{g}}$

From (1) and (2) we know that $\underline{u|_C = f}$ and $\underline{u_x|_C = g}$

Thus from (3) we have

$$\underline{u_y|_C = \dot{f} - g}$$

Now seek $u_{xx}|_C\,,\ u_{xy}|_C\,,\ u_{yy}|_C$:

Need 3 equations.

PDE itself is one equation, (4) is another

The third comes from
$\dfrac{d^2(1)}{d\tau^2} = \dfrac{d(3)}{d\tau} :=$
$u_{xx}\dot{X}^2 + u_{xy}\dot{X}\dot{Y} + u_x\ddot{X} + u_{yx}\dot{Y}\dot{X} + u_{yy}\dot{Y}^2 + u_y\ddot{Y} = \ddot{f}$
or $\underline{u_x|_C + 2\,u_{xy}|_C + u_{yy}|_C = \ddot{f}}$ (5)

Thus we have (dropping $|_C$ but remembering it's still there) on C:

$$\begin{cases} u_{xx} + 2u_{xy} + u_{yy} = \ddot{f} \\ u_{xx} + u_{xy} = \dot{g} \\ u_{xx} + 8u_{xy} + 7u_{yy} = 0 \end{cases}$$

or

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 8 & 7 \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} \ddot{f} \\ \ddot{g} \\ 0 \end{pmatrix}$$

To solve this we must check first that
$\triangle = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 8 & 7 \end{vmatrix} \neq 0$. However
$\triangle = 1\begin{vmatrix} 1 & 0 \\ 8 & 7 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 1 & 7 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 8 \end{vmatrix} = 1 \times 7 - 2 \times 7 + 1 \times (8-1) = \underline{0}$

Therefore no solution <u>exists</u> on C.

Hence cannot find Taylor expansion about C. Hence cannot find any solution with this boundary data. The reason is that $(x, y) = (\tau, \tau)$ is a characteristic curve, cf (i), where $y = x + d$ are the characteristics. Consequently C is more than just tangential to a characteristic!

(iii) Now given data on $C : (\tau, 0)$

$$\begin{array}{rclrcl} X(\tau) & = & \tau & Y(\tau) & = & 0 \\ \dot{X} & = & 1 & \dot{Y} & = & 0 \quad (\star) \\ \ddot{X} & = & 0 & \ddot{Y} & = & 0 \end{array}$$

Given boundary data:

$$\begin{array}{rcll} u(X,Y) & = & f(\tau) & (6) \\ u_y(X,Y) & = & g(\tau) & (7) \end{array}$$

$$\begin{array}{rcccccl} \dfrac{d(6)}{d\tau} := & u_x\dot{X} & + & u_y\dot{Y} & = & \dot{f} \Rightarrow \underline{u_x|_C = \dot{f}} & (8) \\ \dfrac{d(7)}{d\tau} := & u_{yx}\dot{X} & + & u_{yy}\dot{Y} & = & \dot{g} \Rightarrow \underline{u_{xy}|_C = \dot{g}} & (9) \end{array}$$

Thus $\underline{u|_C = f,\ u_x|_C = \dot{f},\ u_y|_C = g}$

Need to find $u_{xx}|_C,\ u_{xy}|_C,\ u_{yy}|_C$:

Have 2 equations from PDE itself and (9). Get 3rd from

$\dfrac{d(8)}{d\tau} :=$

$u_{xx}\dot{X}^2 + u_{xy}\dot{X}\dot{Y} + u_x\ddot{X} + u_{yx}\dot{Y}\dot{X} + u_{yy}\dot{Y}^2 + u_y\ddot{Y} = \ddot{f}$

$\Rightarrow \underline{u_{xx} = \ddot{f}}$

Thus we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 8 & 7 \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{yx} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} \ddot{f} \\ \dot{g} \\ 0 \end{pmatrix}$$

$$\Rightarrow \left\{ \begin{array}{rcl} u_{xx} & = & \ddot{f} \\ u_{xy} & = & \dot{g} \\ u_{yy} & = & -\dfrac{(\ddot{f} + 8\dot{g}}{7} \end{array} \right.$$

Thus expansion of solution about $(0,0)$ is given by expansion in part (ii) for the appropriate value of τ.

What is it? $(x_0, y_0) = (0,0) \equiv \underline{\tau = 0}$ from $(\star)$

Thus

$$\begin{aligned} u(x,y) &= f(0) + x\dot{f}(0) + yg(0) \\ &\quad + \frac{1}{2}x^2\ddot{f}(0) + xy\dot{g}(0) - \left(\frac{\ddot{f}(0) + 8\ddot{g}(0)}{14}\right) y^2 + \cdots \end{aligned}$$