Question
Consider the two dimensional wave equation,

$$c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

with Cauchy boundary conditions:

$$u(x,0) = f(x),\ u_y(x,0) = g(x0.$$

Recall from lectures that the general solution is given by

$$u(x,y) = \frac{1}{2}[f(x+cy) + f(x-cy)] - \frac{1}{2c}\int_{x-cy}^{x+cy} g(s)ds.$$

Assuming that f and g are suitably differentiable, calculate the Taylor expansion of this solution about $9x, y) = (0,0)$ up to second order,

(i) directly from the exact solution

(ii) by parametrising the given boundary conditions on given curve C : $(x,y) = (X(\tau), Y(\tau)) = (\tau, 0)$.

Confirm that the two agree.

Answer
$c^2 u_{xx} - u_{yy} = 0$
(1) $u(x,0) \ = f(x)$
(2) $u_y(x,0) = g(x)$
$\Rightarrow X(\tau) = \tau,\ Y(\tau) = 0$
Data is given on x-axis. Hence C is parametrised as:
On $C : (x,y) = (X(\tau), Y(\tau))$

$$(3) \Rightarrow \begin{cases} \dot{X} = 1,\ \dot{Y} = 0 \\ \ddot{X} = 0,\ \ddot{Y} = 0 \end{cases}$$

(i) Must expand exact solution as Taylor about $(0,0)$

$$u(x,y) = \frac{1}{2}[f'(x+cy) + f(x-cy)] + \frac{1}{2c}\int_{x-cy} x + cyg(s)\,ds$$

$$\begin{cases} \dfrac{\partial u}{\partial x} &= \dfrac{1}{2}[f' + f'] + \dfrac{1}{2c}[g(x+cy) - g(x-cy)] \\ \Rightarrow \left.\dfrac{\partial u}{\partial x}\right|_{(0,0)} &= \dfrac{1}{2}(f'(0) + f'(0)] + \dfrac{1}{2c}[g(0) - g(0)] = \underline{f'(0)} \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial y} &= \dfrac{1}{2}[cf' - cf'] + \dfrac{c}{2c}[g(x+cy)+g(x-cy)] \\ \Rightarrow \left.\dfrac{\partial u}{\partial y}\right|_{(0,0)} &= \dfrac{1}{2}(cf'(0) - cf'(0)] + \dfrac{1}{2}[g(0)+g(0)] = \underline{g(0)} \end{cases}$$

Also $u(0,0) = \dfrac{1}{2}[f(0)+f(0)] + \dfrac{1}{2c}\displaystyle\int_0^0 g(s)\,ds = \underline{f(0)}$

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{2}[f" + f"] + \dfrac{1}{2c}[g' - g'] \Rightarrow \left.\dfrac{\partial^2 u}{\partial x^2}\right|_{(0,0)} = f"(0)$

$\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{1}{2}[cf" - cf"] + \dfrac{1}{2c}[cg' + cg'] \Rightarrow \left.\dfrac{\partial^2 u}{\partial x \partial y}\right|_{(0,0)} = g'(0)$

$\dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{2}[c^2 f" + c^2 f"] + \dfrac{1}{2c}[c^2 g' - c^2 g'] \Rightarrow \left.\dfrac{\partial^2 u}{\partial y^2}\right|_{(0,0)} = c^2 f"(0)$

Thus the Taylor expansion around $(0,0)$ is

$$\begin{aligned} u(x,y) &= u(0,0) + x\left.\frac{\partial u}{\partial x}\right|_{(0,0)} + y\left.\frac{\partial u}{\partial y}\right|_{(0,0)} \\ &\quad + \frac{x^2}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_{(0,0)} + xy\left.\frac{\partial^2 u}{\partial x \partial y}\right|_{(0,0)} + \frac{y^2}{2}\left.\frac{\partial^2 u}{\partial y^2}\right|_{(0,0)} + \cdots \end{aligned}$$

$$\begin{aligned} u(x,y) &= f(0) + xf'(0) + yg(0) \\ &\quad + \frac{x^2}{2} f"(0) + xyg'(0) + \frac{y^2 c^2}{2} f"(0) + \cdots \end{aligned}$$

(ii) Seek $u|_{(0,0)}$, $u_x|_{(0,0)}$, $u_y|_{(0,0)}$, $u_{xx}|_{(0,0)}$, $u_{xy}|_{(0,0)}$, $u_{yy}|_{(0,0)}$ from equation and boundary data only.

From (1) $\quad u(0,0) = f(0)$

From (2) $\quad u_y(0,0) = g(0)$

Get u_x by differentiating (1) with respect to τ:

$\dfrac{d(1)}{d\tau} := u_x|_C \dot{X} + u_y|_C \dot{Y} = f_x \dot{X} \Rightarrow \underline{u_x|_C = f'(x)}$

Now seek $u_{xx}|_{(0,0)}$, $u_{xy}|_{(0,0)}$, $u_{yy}|_{(0,0)}$ etc. from 3 equations.

First is PDE itself:

(4) $\underline{x^2\, u_{xx}|_C - u_{yy}|_C = 0}$

Second is

$\frac{d(2)}{d\tau} := u_{yx}|_C \dot{X} + u_{yy}|_C \dot{Y} = g_x \dot{X} \Rightarrow \underline{u_{xy}|_C = g'(x)}$ from (3)

Third is from

$\frac{d^2(1)}{d\tau^2} :=$

$u_{xx}|_C \dot{X}^2 + u_{xy}|_C \dot{X}\dot{Y} + u_x|_C \ddot{X}$

$+ u_{yx}|_C \dot{Y}\dot{X} + u_{yy}|_C \dot{Y}^2 + u_y|_C \ddot{Y} = f_{xx}\dot{X}^2 + f_x\ddot{X}$

$\Rightarrow \underline{u_{xx}|_C = f_{xx} = f"(x)}$ from (3)

Thus we have

$$\begin{array}{ccc} (7) & \rightarrow \\ (6) & \rightarrow \\ (5) & \rightarrow \end{array} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c^2 & 0 & -1 \end{pmatrix}} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} f" \\ g'(x) \\ 0 \end{pmatrix}$$

So $\triangle = det() = -1 \neq$ anywhere on C so can find solution

Not difficult to solve

$$\Rightarrow \left\{ \begin{array}{lcl} u_{xx} & = & f''(x) \\ u_{xy} & = & g'(x) \\ u_{yy} & = & c^2 f''(x) \end{array} \right\} (B)$$

Hence expanding about $(0,0)$ we have:

$$\begin{aligned} u(x,y) &= f(0) \text{ (from (1) } + \text{xf}'(0) \text{ (from (A) } + \text{yg}(0) \text{ (from (2)} \\ &\quad + \underbrace{\frac{x^2}{2} f''(0) + xyg'(0) + \frac{y^2c^2}{2} f''(0)} + \cdots \end{aligned}$$

which is exactly the same as the result of part (i) as required.