Question

Find the region in which the equation

$$[(x-y)^2-1]u_{xx}+2uxy+[(x-y)^2-1]u_{yy}=0$$

is of hyperbolic type. Write down the differential equation of the characteristics and, by making the change of variables $\xi=x+y,\ \eta=x-y$, show that it may be written as

$$\frac{d\xi}{d\eta}=\pm\frac{eta}{\sqrt{2-\eta^2}}.$$

Deduce the form of the characteristics.

Answer

$a=[(x-y)^2-1],\ b=1,\ c=[(x-y)^2-1]$

$b^2-ac=1-[(x-y)^2-1]^2=(x-y)^2(2-(x-y)^2)$

so is hyperbolic for $0<(x-y)^2<2$

i.e., $0<|x-y|<\sqrt{2}$

Parabolic on $x=y$ i.e., $o=|x-y|,\ \sqrt{2}=|x-y|,\ (x-y)=\sqrt{2}$

elliptic for $(x-y)^2.2$ i.e., $\sqrt{2}<|x-y|$

Characteristics when parabolic/hyperbolic given by

$$[(x-y)^2-1]dx^2-2dxdy+[(x-y)^2-1]dy^2=0\quad(A)$$

or $\underline{\dfrac{dy}{dx}=\dfrac{2\pm\sqrt{(x-y)^2(2-(x-y)^2)}}{[(x-y)^2-1]}}$ <u>not nice to solve</u>

Make change $\left\{\begin{array}{c}\xi=x+y\\ \eta=x-y\end{array}\right\}$ as per question

So

$$\left\{\begin{array}{l}x=\frac{1}{2}(\xi+\eta)\\ y=\frac{!}{2}9\xi-\eta)\end{array}\right\}\Rightarrow\left\{\begin{array}{l}dx=\frac{1}{2}(d\xi+d\eta)\\ dy=\frac{1}{2}(d\xi-d\eta)\end{array}\right\}$$

Substitute this into (A):

$$\frac{(\eta^2-1)}{4}(d\xi+d\eta)^2-\frac{1}{2(d\xi^2-d\eta^2)}+\frac{(\eta^2-1)}{4}(d\xi-d\eta)^2=0$$

$$\Rightarrow[(\eta-1)^2-1]d\xi^2+[(\eta^2-1)+1]d\eta^2=0$$

$$\Rightarrow\frac{d\xi}{d\eta}=\pm\sqrt{-\left[\frac{(\eta^2-1)+1}{(\eta^2-1)-1}\right]}=\pm\frac{\eta}{(2-\eta^2)^{\frac{1}{2}}}\text{ as required}$$

Solve this for η

$$\begin{aligned}\int d\xi &= \pm\int \frac{d\eta\eta}{(2-\eta^2)^{\frac{1}{2}}} \\ \xi &= \mp\sqrt{2-\eta^2} + const\end{aligned}$$

or

$$\underline{(\xi + c)^2 + \eta^2 = 2}$$

Circles centred at $\xi = -c$ radius $\sqrt{2}$
In original coords

$$(x + y + c)^2 + (x - y)^2 = 2$$

or see diagram below:
PICTURE