QUESTION
Let W_t and $\tilde{W}_t$ be two independent Brownian motions and p a constant between -1 and 1. Is the process $X_t = pW_t + \tilde{W}_t\sqrt{1-p^2}$ continuous? What is its distribution? Is X_t a Brownian motion?
ANSWER

$$X_t = PW_t + \tilde{W}_t(1-\rho^2)^{\frac{1}{2}}$$

The process is continuous since W_t, $\tilde{W}_t$ and ρ are.
Mean:

$$\begin{aligned}\langle X_t\rangle &= \left\langle PW_t + \tilde{W}_t(1-p^2)^{\frac{1}{2}}\right\rangle \\ &= \rho\langle W_t\rangle + (1-p^2)^{\frac{1}{2}}\left\langle \tilde{W}_t\right\rangle \\ &= 0\end{aligned}$$

since $W_t \sim N(0,t)$ and $\tilde{W}_t \sim N(0,t)$ by definition.
Variance:

$$\begin{aligned}\left\langle (X_t-0)^2\right\rangle &= \left\langle (PW_t + \tilde{W}_t(1-\rho^2)^{\frac{1}{2}})^2\right\rangle \\ &= \left\langle \rho^2 W_t^2 + \tilde{W}_t^2(1-\rho^2) + 2PW_t\tilde{W}_t(1-p^2)^{\frac{1}{2}}\right\rangle \\ &= \rho^2\left\langle W_t^2\right\rangle + (1-\rho^2)\left\langle \tilde{W}_t^2\right\rangle + 2\rho\sqrt{1-\rho}\left\langle W_t\tilde{W}_t\right\rangle\end{aligned}$$

Now if W_t and $\tilde{W}_t$ are independent they are uncorrelated, thus $\left\langle W_t\tilde{W}_t\right\rangle = 0$.
Hence

$$\begin{aligned}\left\langle (X_t-0)^2\right\rangle &= \rho^2\sigma_t^2 + (1-\rho^2)\tilde{\sigma}_t^2 \\ &= \rho^2 t + (1-\rho^2)t \\ &= t\end{aligned}$$

Where σ_t^2 is the variance of W_t^2 (mean is zero) and $\tilde{\sigma}_t^2$ is the variance of $\tilde{W}_t^2$ (mean is zero).
Thus since W_T and $\tilde{W}_t$ are normal, $X_t \in N(0,t)$
Is it Brownian? Check conditions:

(i) X_t is continuous and $X_0 = 0$: It is continuous and by definition W_0 and $\tilde{W}_0$ are zero so condition is satisfied.

(ii) $X_t \in N(0,t)$

(iii) $X_{s+t} - X_t \in N(0,t)$ and independent of time $< s$:

$$X_{s+t} - X_s \in \rho \underbrace{\underbrace{(W_{t+s} - W_s)}_{\in N(0,t)} + \sqrt{1-\rho^2}\underbrace{(\tilde{W}_{t+s} - \tilde{W}_s)}_{\in N(0,t)}}_{\text{by definition if } W_t,\ \tilde{W}_t \text{ are brownian}}$$

Thus from the above calculations we have

$$X_{s+t} - X_s \in N(0,t)$$

The W_S and $\tilde{W}_s$ are independent of S by definition too, so $X_{s+t} - X_s$ must be independent of S. So (iii) is satisfied.

Thus X_t is Brownian.