QUESTION Discrete random variables X and Y have the joint probability function given in the following table.

Y\X	1	2	3
1	0.1	0.2	0
2	0.1	0.1	0.1
3	0.2	0	0.2

(i) Calculate E(X), E(Y), Var(X), Var(Y), Cov(X,Y).

(ii) Tabulate the probability function of Z=2Y-X and use these calculations to find E(Z) and Var(Z). Check your values using the results obtained in(i).

(iii) (X_1, Y_1) and (X_2, Y_2) are independent pairs of random variables where each pair has the joint probability function given above. Calculate $P(X_1 + Y_1 = 2(X_2 + Y_2))$.

ANSWER

Y\X	1	2	3	$P_Y(y)$
1	0.1	0.2	0	0.3
2	0.1	0.1	0.1	0.3
3	0.2	0	0.2	0.4
$P_X(x)$	0.4	0.3	0.3	

(i) $E(X) = 0.4 \times 1 + 0.3 \times 2 + 0.3 \times 3 = 1.9$
$E(Y) = 0.3 \times 1 + 0.3 \times 2 + 0.4 \times 3 = 2.1$
$E(X^2) = 0.4 \times 1^2 + 0.3 \times 2^2 + 0.3 \times 3^2 = 4.3$
$Var(X) = 4.3 - 1.9^2 = 0.69$
$E(Y^2) = 0.3 \times 1^2 + 0.3 \times 2^2 + 0.4 \times 3^2 = 5.1$
$Var(Y) = 5.1 - 2,1^2 = 0.69$
$E(XY) = 0.1 \times 1 + 0.2 \times 2 + 0 \times +0.1 \times 2 + 0.1 \times 4 + 0.1 \times 6 + 0.2 \times 3 + 0 \times 6 + 0.2 \times 9 = 4.1$
$Cov(X, Y) = 4.1 - 1.9 \times 2.1 = 0.11$

(ii) Z=2Y-X takes values from 0(Y=1,X=2) to 5(Y=3,X=1)

Z	0	1	2	3	4	5
Y=1	0.2	0.1				
Y=2		0.1	0.1	0.1		
Y=3				0.2		0.2
p(z)	0.2	0.2	0.1	0.3		0.2

$E(Z) = 1 \times 0.2 + 2 \times 0.1 + 3 \times 0.3 + 5 \times 0.2 = 2.3$
$E(2Y - Z) = 2E(Y) - E(X) = 4.2 - 1.9 = 2.3$
$E(Z^2) = 1^2 \times 0.2 + 2^2 \times 0.1 + 3^2 \times 0.3 + 5^2 \times 0.2 = 8.3$

$\text{Var}(Z) = 8.3 - 2.3^2 = 3.01$

$$\begin{aligned} \text{Var(2Y-Z)} &= 4\text{Var(Y)}+\text{Var(X)}-4\text{Cov(X,Y)} \\ &= 4 \times 0.69 + 0.69 - 4 \times 0.11 = 3.01 \end{aligned}$$

(iii) $X_1 + Y_1$ and $X_2 + Y_2$ take values 2,3,4,5,6.
$P(X_1 + Y_1 = 2(X_2 + Y_2))$

$$\begin{aligned} &= P(X_1 + Y_1 = 2, X_2 + Y_2 = 4 \text{ or } X_1 + Y_1 = 3, X_2 + Y_2 = 6) \\ &= P(X_+Y_1 = 2)P(X_2 + Y_2 = 4) + P(X_1 + Y_1 = 3)P(X_2 + Y_2 = 6) \\ &= P(X_1 = 1, Y_1 = 1)P \begin{pmatrix} X_2 = 1, Y_2 = 3 \\ X_2 = 2, Y_2 = 2 \\ X_2 = 3, Y_2 = 1 \end{pmatrix} \\ &+ P \begin{pmatrix} X_1 = 1, Y_1 = 2 \\ X_1 = 2, Y_1 = 1 \end{pmatrix} P(2 = 3, Y_2 = 3) \\ &= 0.1 \times 0.3 + 0.3 \times 0.2 = 0.9 \end{aligned}$$