QUESTION A random variable X has pdf $f(X) = \frac{k}{2}e^{-x}(1+x)^2,\ -1 \leq x \leq \infty$. Find k. Show that the mgf of X is $\frac{e^{-t}}{(1-t)^3},\ t < 1$. Hence find Var(X). By extending the results in an obvious way find also $E(X-\mu)^3$.

ANSWER $f(x) = \frac{k}{2}e^{-x}(1+x)^2 \quad -1 \leq x \leq \infty$

$$\begin{aligned} M(t) &= \frac{k}{2}\int_{-} 1^{\infty} e^{xt}e^{-x}(1+x)^2\,dx \quad \text{let } y = 1+x \\ &= \frac{k}{2}\int_0^{\infty} e^{(y-1)t}e^{-(y-1}y^2\,dy \\ &= e^(1-t)\frac{k}{2}\int_0^{\infty} e^{-y(1-t)}y^2\,dy \end{aligned}$$

(from $\varrho(\lambda)$ we know $\int_0^{\infty} \lambda e^{-\lambda x}x^2\,dx = E(X^2) = \frac{2}{\lambda^2}$. Let $\lambda = 1-t$ or integrate by parts.)

$$\begin{aligned} M(t) &= \frac{k}{2}e^{(1-t)}\frac{2}{(1-t)^3} \quad t < 1 \\ &= \frac{ke^(1-t)}{(1-t)^3} \\ M(0) &= 1 \text{ therefore } 1 = ke \text{ therefore } k = \frac{1}{e} \\ M(t) &= \frac{e^{-t}}{(1-t)^3} \\ &= (1-t+\ldots)(!+3t+\ldots) \\ &= 1+2t+\ldots \end{aligned}$$

μ=coefficient of t=2

$$\begin{aligned} M*(t) &= \frac{e^{-3t}}{(1-t)^3} \\ &= (1-3t+\frac{0t^2}{2}-\frac{27t^3}{3!}+\ldots)(1+3t+6t^2+10t^3+\ldots) \\ &= 1+t^2(\frac{9}{2}-9+6)+t^3(-\frac{27}{6}+\frac{27}{2}-18+10)+\ldots \\ &= 1+\frac{3t^2}{2}+t^3+\ldots \end{aligned}$$

$\sigma^2 = 2!\times$ the coefficient of $t^2 = 3$ $E(X-\mu)^3 = 3!\times$ the coefficient of $t^3 = 6$
Alternately differentiate, $\mu = M^{(1)}(0),\ \sigma^2 = M^{*(2)}(0)$ $E(X-\mu)^3 = M^{*(2)}(0)$
or extend $\sigma^2 = E(X^2) - \mu^2$
$E(X-\mu)^3 = E(X^3) - 3\mu E(X^2) + 2\mu^3$ and use M(t) throughout.