QUESTION Find the mgf of the square of a standard normal variable. Hence find the mgf of a χ^2-distribution with ν degrees of freedom. Hence show that the χ^2-distribution is a particular form of a gamma distribution and state the parameters of gamma. What form does the χ^2-distribution take if ν=2.

ANSWER

$$\begin{aligned} M_{x^2}(t) &= E(e^{X^2 t}) \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} e^{x^2 t}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2(1-2t)}\, dx \\ &= \frac{1}{(1-2t)^{\frac{1}{2}}} \end{aligned}$$

(from $N(\mu, \sigma^2)$ we know that $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}\, dx = 1$. let $\mu = 0$, $\sigma^2 = \frac{1}{1-2t}$).

$M_\nu^2 = X_1^2 + X_2^2 + \ldots + X_\nu^2$ where each X_i has an independent N(0,1) distribution hence $M_{X_\nu^2}(t) = [M_{X^2}(t)]^\nu = \frac{1}{(1-2t)^{\frac{\nu}{2}}}$

For$\varrho(\lambda)$ $M(t) = \frac{\lambda}{\lambda - t} = \frac{1}{1-\frac{t}{\lambda}}$

For Gamma m, λ $M(t) = (\frac{1}{1-\frac{t}{\lambda}})^2$(this follows for integer m and in fact works for a general m)

Hence by uniqueness X_ν^2 is Gamma $m = \nu$ $\lambda = \frac{1}{2}$. If $\nu = 2$ $M(t) = \frac{1}{1-2t}$ i.e. $\varrho(\frac{1}{2})$