Question

Find the co-ordinates of the point of intersection of the tangents at the points with parameters t_1, t_2 of the parabola $x = kt^2$, $y = 2kt$. Prove that the tangents intersect at right angles if and only they intersect on the directrix.

Answer

$x = kt^2 \quad y = 2kt \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2k}{2kt} = \frac{1}{t}$

So the equation of the tangent is

$$\begin{aligned} y - 2kt &= \frac{1}{t}(x - kt^2) \\ yt &= x + kt^2 \end{aligned}$$

So $yt_1 = x + kt^2$ and $yt_2 = xkt_2^2$ intersect where $y(t_1 - t_2) = k(t_1^2 - t_2^2)$

i.e. where $y = k(t1 + t2)$

So $x = kt_1(t_1 + t_2) - kt_1^2 = kt_1t_2$

The directrix has equation $x = -k$

So the intersection lies on the directirx if and only if $t_1t_2 = -1$ i.e. $\frac{1}{t_1} \cdot \frac{1}{t_2} = -1$

i.e. if the tangents intersect orthogonally