Question
Verify that the equation

$$r = \frac{a}{\sin^2 \frac{1}{2}\theta}$$

is the polar equation of a parabola. Prove that $\phi = \pi - \frac{1}{2}\theta$ and use this to deduce the parabolic mirror property.

Answer
$r = \frac{a}{\sin^2 \frac{1}{2}\theta} = 2a1 - \cos\theta$
So $\frac{2a}{r} = 1 - \cos\theta$. This is the standard equation of a conic with eccentricity 1. i.e. a parabola

$$\begin{aligned} \cot\phi &= \frac{1}{r}\frac{dr}{d\theta} \\ &= \frac{1}{r}a\left(-2\left(\sin\frac{1}{2}\theta\right)^{-3}\right)\frac{1}{2}\cos\frac{1}{2}\theta \\ &= -\cot\frac{1}{2}\theta \\ \text{So} \quad \phi &= \pi - \frac{1}{2}\theta \end{aligned}$$