Question

If c is a fixed positive constant, then

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2 - c^2} = 1 \qquad c^2 < \lambda^2$$

defines a family of ellipses, any member of which is characterised by a particular value of λ. Show that every member of the family

$$\frac{x^2}{\mu^2} - \frac{y^2}{c^2 - \mu^2} = 1 \qquad \mu^2 < c^2$$

intersects any member of the first family at right angles.

Answer

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ gives $\dfrac{dy}{dx} = -\dfrac{x}{y} \cdot \dfrac{b^2}{a^2}$

$\dfrac{x^2}{c^2} - \dfrac{y^2}{d^2} = 1$ gives $\dfrac{dy}{dx} = \dfrac{x}{y} \cdot \dfrac{d^2}{c^2}$

So the product of slopes where a curve of each system intersects is

$$-\frac{x^2}{y^2} \cdot \frac{\lambda^2 - c^2}{\lambda^2} \cdot \frac{c^2 - \mu^2}{\mu^2}$$

where they intersect

$$-\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2 - c^2} = 1 = \frac{x^2}{\mu^2} - \frac{y^2}{c^2 - mu^2}$$

So $\dfrac{x^2}{y^2}\dfrac{(\lambda^2 - c^2)(c^2 - \mu^2)}{\lambda^2\mu^2} = 1$

Thus the systems of curves are mutually orthogonal.

The foci are all at $(\pm c, 0)$