

Question

Find a condition on l, m, n to ensure that the line $lx + my + n = 0$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Answer

The equation of the tangent at (x_0, y_0) is

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$$

This is the same as the line $-\frac{l}{n}x - \frac{m}{n}y = 1$ if and only if

$$\frac{x_0}{a^2} = -\frac{l}{n} \quad \frac{y_0}{b^2} = -\frac{m}{n}$$

$$\text{i.e. } x_0 = -\frac{l}{a^2}n \quad y_0 = -\frac{mb^2}{n}$$

But (x_0, y_0) lies on the ellipse, so

$$1 = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = \frac{l^2 a^4}{n^2 a^2} + \frac{m^2 b^2}{n^2 b^2} \quad \text{i.e. } l^2 a^2 + m^2 b^2 = n^2$$