Question

(a) Show that the force field $\mathbf{F} = -\kappa r^3 \mathbf{r}$ is conservative.

(b) What is the potential energy of this force field?

(c) If a particle of mass m moves with velocity $\mathbf{v} = \dot{\mathbf{r}}$ in this force field, show that if E is the constant total energy then

$$\frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \frac{1}{5} r^5 = E.$$

What important physical principle does this illustrate?

Answer

(a)

$$\begin{aligned} \mathbf{F} &= -\kappa r^3 \mathbf{r} \\ &= -\kappa r^3 (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \quad \text{where } \ r = \sqrt{x^2 + y^2 + z^2} \end{aligned}$$

$$\begin{aligned} \nabla \times \mathbf{F} &= -\kappa \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^3 & yr^3 & zr^3 \end{vmatrix} \\ &= -\kappa \left\{ \mathbf{i} \left(\frac{\partial}{\partial y}(zr^3) - \frac{\partial}{\partial y}(yr^3) \right) + ... \right\} \\ &= -\kappa \left\{ \mathbf{i} 3r^2 \left(z\frac{\partial r}{\partial y} - y\frac{\partial r}{\partial z} \right) + ... \right\} \end{aligned}$$

Now $\dfrac{\partial r}{\partial z} = \dfrac{z}{r} \quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ therefore $z\dfrac{\partial r}{\partial z} - y\dfrac{\partial r}{\partial y} = 0$

Therefore the $\mathbf{i}$ component of $\nabla \times \mathbf{F} = 0$, as are the $\mathbf{j}$ and $\mathbf{k}$ components by symmetry.

Therefore $\nabla \times \mathbf{F} = 0$.

(b) Potential energy. $\mathbf{U} = -\int \mathbf{F} \cdot d\mathbf{r}$ for all paths.

As $\mathbf{F}$ is conservative, choose a radial path.

Therefore

$$\begin{aligned} \mathbf{U} &= -\int -\kappa r^3 \mathbf{r} \frac{d\mathbf{r}}{dt} dt \\ &= \kappa \int r^4 dr \\ &= \frac{\kappa}{5} r^5 (+\text{constant}) \end{aligned}$$

(c) K.E. + P.E. = constant

Therefore $\frac{1}{2} m \dot{\mathbf{r}}^2 + \frac{\kappa}{5} r^5 = E$

"Conservation of Energy"