Question
Find the work done in moving a particle in the force field

$$\mathbf{F} = 3x^2\mathbf{i} + (2xz - y)\mathbf{j} + z\mathbf{k}$$

(a) along the straight line from (0,0,0) to (2,1,3),

(b) along the space curve $x = 2t^2$, $y = t$, $z = 4t^2 - t$, from $t = 0$ to $t = 1$.

(c) Is the work done independent of the path? Explain.

Answer

(a) Straight line $x = 2t$, $y = t$, $z = 3t$ $0 \leq t \leq 1$

$$\begin{aligned}
\text{Work done} &= \int \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^1 \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \\
&= \int_0^1 [3(2t)^2\mathbf{i} + (2.2t.3t - t)\mathbf{j} + 3t\mathbf{k}] \cdot [2\mathbf{i} + \mathbf{j} + 3\mathbf{k}] dt \\
&= \int_0^1 [24t^2 + 12t^2 - t + 9t] dt \\
&= \left[8t^3 + 4t^3 \frac{1}{2}t^2 + \frac{9}{2}t^2\right]_0^1 \\
&= 8 + 4 + 4 \\
&= 16
\end{aligned}$$

(b)

$$\begin{aligned}
\text{Work done} &= \int \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^1 \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \\
&= \int_0^1 \left[3((2t)^2)^2\mathbf{i} + (2.2t^2(4t^2 - t).(-t))\mathbf{j} + (4t^2 - t)\mathbf{k}\right] \\
&\quad \cdot [4t\mathbf{i} + \mathbf{j} + (8t - 1)\mathbf{k}] dt \\
&= \int_0^1 \left[48t^5 + 16t^4 - 4t^3 - t + (4t^t - t)(8t - 1)\right] dt \\
&= \left[8t^6 + \frac{16}{5}t^5 - t^4 - \frac{1}{2}t^2 + \frac{1}{2}(4t^2 - t)^2\right]_0^1
\end{aligned}$$

$$
\begin{aligned}
&= 8 + \frac{16}{5} - 1 - \frac{1}{2} + \frac{9}{2} \\
&= 14\frac{1}{5}
\end{aligned}
$$

(c) No. The force is not conservative.