Question

Suppose that X and Y is $N_2(\mu_x = 0, \mu_y = 0, \sigma_x = 1, \sigma_y = 1, \rho)$. Show that the sum $X + Y$ and the difference $X - Y$ are independent random variables. (Hint: Use the transformation technique.)

Answer

Let $U = X + Y,\ V = X - Y$

Find the joint pdf of U and V where the joint pdf of X and Y is

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho)^2}\{x^2 - 2\rho xy + y^2\}},$$

$$(-\infty < x < \infty,\ -\infty < y < \infty)$$

$x = \frac{u+v}{2}; \quad y = \frac{u-v}{2}$

$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

Now

$$\begin{aligned} & x^2 - 2\rho xy + y^2 \\ = & \left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2 - 2\rho\frac{u+v}{2}\frac{u-v}{2} \\ = & \frac{u^2}{2}(1-\rho) + \frac{v^2}{2}(1+\rho) \text{ (after algebra)} \end{aligned}$$

Therefore

$$\begin{aligned} f_{U,V}(u, v) &= \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho)^2}\left\{\frac{u^2}{2}(1-\rho) + \frac{v^2}{2}(1-\rho)\right\}} \left|-\frac{1}{2}\right| \\ &= \frac{1}{4\pi\sqrt{1-\rho^2}}\, e^{-\frac{u^2}{4(1+\rho)} - \frac{v^2}{4(1+\rho)}} \\ & \quad -\infty < u < \infty,\ -\infty < v < \infty \end{aligned}$$

Easy to see that $U \sim N\{0, \sigma^2 = 2(1+\rho)\}$ and $V \sim N\{0, \sigma^2 = 2(1-\rho)\}$ and U and V are independent.