Question

For each subset S of $\mathbf{R}$ given below, determine whether S is bounded above, bounded below, bounded, or neither. If S is bounded above, determine $\sup(S)$, and decide whether or not $\sup(S)$ is an element of S. If S is bounded below, determine $\inf(S)$, and decide whether or not $\inf(S)$ is an element of S.

1. $S = \left\{ \frac{1}{n} \mid n \in \mathbf{Z} - \{0\} \right\}$;
2. $S = \{2^x \mid x \in \mathbf{Z}\}$;
3. $S = [-1, 1] \cup \{5\} = \{x \in \mathbf{R} \mid -1 \leq x \leq 1\} \cup \{5\}$;
4. $S = \left\{ \frac{x}{2^y} \mid x,\ y \in \mathbf{N} \right\}$;
5. $S = \left\{ \frac{n+1}{n} \mid n \in \mathbf{N} \right\}$;
6. $S = \left\{ (-1)^n \left(1 + \frac{1}{n}\right) \mid n \in \mathbf{N} \right\}$;
7. $S = \{x \in \mathbf{Q} \mid x^2 < 10\}$;
8. $S = \{x \in \mathbf{R} \mid |x| > 2\}$;

Answer

1. Bounded above by 1 (since for $n \in \mathbf{Z} - \{0\}$, either $n \geq 1$ in which case $\frac{1}{n} \leq 1$, or $n \leq -1$, in which case $\frac{1}{n} \leq 0$), and so has a supremum. Since 1 is an upper bound for S and since $1 \in S$, $1 = \sup(S)$. In this case, $\sup(S) \in S$.

   Bounded below by -1 (since for $n \in \mathbf{Z} - \{0\}$, either $n \geq 1$, in which case $0 < \frac{1}{n}$, or $n \leq -1$, in which case $\frac{1}{n} \geq \frac{1}{-1} = -1$), and so has an infimum. Since -1 is a lower bound for S and since $-1 \in S$, $-1 = \inf(S)$. In this case, $\inf(S) \in S$.

   Since S is both bounded above and bounded below, it is bounded.

2. Bounded below by 0 (since $2^x > 0$ for all $x \in \mathbf{R}$, we certainly have that $2^x > 0$ for all $x \in \mathbf{Z}$), and so has an infimum. Given any $\varepsilon > 0$, we can always find x so that $2^x < \varepsilon$, namely take $\log_2$ of both sides, and take x to be any integer less than $\log_2(\varepsilon)$. Hence, there is no positive lower bound, and so the greatest lower bound, the infimum, is $\inf(S) = 0$. Since there are no solutions to $2^x = 0$, in this case $\inf(S) \notin S$.

Since $2^x > x$ for positive integers x, given any $C > 0$ we can find an x so that $2^x > C$, and so there is no upper bound. That is, S is not bounded above.

Since S is not bounded above, it is not bounded.

3. Bounded below by -1 (since $[-1, 1] = \{x \in \mathbf{R} \mid -1 \leq x \leq 1\}$ and since $-1 < 5$), and so has an infimum. Since -1 is a lower bound for S and since $-1 \in S$, $-1 = \inf(S)$. In this case, $\inf(S) \in S$.

   Bounded above by 5, and so has a supremum. Since 5 is an upper bound for S and since $5 \in S$, $5 = \sup(S)$. In this case, $\sup(S) \in S$.

   Since S is both bounded above and bounded below, it is bounded.

4. Considering the subset of S in which $y = 1$, we have that S contains the natural numbers $\mathbf{N}$, and hence S is not bounded above.

   Since x and 2^y are both positive for $x, y \in \mathbf{N}$, we have that $\frac{x}{2^y} > 0$ for all $x, y \in \mathbf{N}$. Therefore, S is bounded below by 0, and so has an infimum. Considering the subset of S in which $x = 1$, we have that S contains $\frac{1}{2^y}$ for all $y \in \mathbf{N}$. In particular, for each $\varepsilon > 0$, we can find $y \in \mathbf{N}$ so that $\frac{1}{2^y} < \varepsilon$, namely take $\log_2$ of both sides to get $-y < \log_2(\varepsilon)$, or equivalently $y > \log_2(\varepsilon)$. Hence, there is no positive lower bound, and so $0 = \inf(S)$. Since $\frac{x}{2^y}$ is never 0 for $x, y > 0$, in this case $\inf(S) \notin S$.

   Since S is not bounded above, it is not bounded.

5. Write $\frac{n+1}{n} = 1 + \frac{1}{n}$. Bounded below by 1, since $\frac{1}{n} > 0$ for all $n \in \mathbf{N}$, and hence $1 + \frac{1}{n} > 1$ for all $n \in \mathbf{N}$. Moreover, since for each $\varepsilon > 1$ we can find n so that $1 + \frac{1}{n} < \varepsilon$, there is no lower bound greater than 1, and so $\inf(S) = 1$. In this case, $\inf(S) \notin S$, since $1 + \frac{1}{n} \neq 1$ for all $n \in \mathbf{N}$.

   Bounded above by 2, since $\frac{1}{n} \leq 1$ for all $n \in \mathbf{N}$ and hence $1 + \frac{1}{n} \leq 2$. In this case, $2 = 1 + \frac{1}{1}$ and so $2 \in S$. Since 2 is an upper bound for S that is contained in S, we have that $2 = \sup(S)$ and so $\sup(S) \in S$.

   Since S is both bounded above and bounded below, it is bounded.

6. Break S up into two subsets, one of the positive terms (when n is even) and the negative terms (when n is odd). So, $S = \{-2, -\frac{4}{3}, -\frac{6}{5}, \ldots\} \cup \{\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \ldots\}$.

   The positive terms are all of the form $1 + \frac{1}{n}$ where n is even. Since $\frac{1}{n}$ decreases as n increases, the largest positive term is $1 + \frac{1}{2} = \frac{3}{2}$, and so

S is bounded above and hence has a supremum. Since S is bounded above by $\frac{3}{2}$ and since $\frac{3}{2} \in S$, $\sup(S) = \frac{3}{2}$, and in this case $\sup(S) \in S$.

The negative terms are all of the form $1 + \frac{1}{n}$ where n is odd. Since $\frac{1}{n}$ decreases as n increases, $-\frac{1}{n}$ increases as n increases, and so the smallest negative term is $\frac{-1-1}{1} = -2$, and so S is bounded below and hence has an infimum. Since S is bounded below by -2 and since $-2 \in S$, $\inf(S) = -2$, and in this case $\inf(S) \in S$.

Since S is both bounded above and bounded below, it is bounded.

7. We can rewrite S as $S = (-\sqrt{10}, \sqrt{10}) \cap \mathbf{Q}$. By the definition of $(-\sqrt{10}, \sqrt{10})$, S is bounded below by $-\sqrt{10}$, and hence has an infimum. Since there are rational numbers greater than $-\sqrt{10}$ but arbitrarily close to $-\sqrt{10}$ (as can be seen by taking the decimal expansion of $-\sqrt{10}$ and truncating it after some number of places to get a rational number near $-\sqrt{10}$), there is no lower bound greater than $-\sqrt{10}$, and so $\inf(S) = -\sqrt{10}$. In this case, $\inf(S) \notin S$.

   S is bounded above by $\sqrt{10}$, and hence has a supremum. Since there are rational numbers less than $\sqrt{10}$ but arbitrarily close to $\sqrt{10}$ (as can be seen by taking the decimal expansion of $\sqrt{10}$ and truncating it after some number of places to get a rational number near $\sqrt{10}$), there is no upper bound less than $\sqrt{10}$, and so $\sup(S) = \sqrt{10}$. In this case, $\sup(S) \notin S$.

   Since S is both bounded above and bounded below, it is bounded.

8. Rewrite S as $S = (-\infty, -2) \cup (2, \infty)$. This set is neither bounded above (since for each real number r, there is $s \in S$ with $s > r$, namely the larger of 3 and $r + 1$) nor bounded below (since for each real number r, there is $s \in S$ with $s < r$, namely the smaller of -3 and $r - 1$).

   Since S is not bounded below, it has no infimum. Since S is not bounded above, it has no supremum.

   Since S is neither bounded above not bounded below, it is not bounded.