Question

In this question, A and B are subsets of $\mathbf{R}$. Show that each of the following holds.

1. $\inf(A \cup B) = \min(\inf(A), \inf(B))$;
2. if $A \cap B \neq \emptyset$, then $\sup(A \cap B) \leq \min(\sup(A), \sup(B))$;
3. if $A \cap B \neq \emptyset$, then $\inf(A \cap B) \geq \max(\inf(A), \inf(B))$;
4. if u is an upper bound for A and if $u \in A$, then $u = \sup(A)$;
5. if t is an lower bound for A and if $t \in A$, then $t = \inf(A)$;
6. if $\inf(A)$ exists, then $\inf(A) = \sup\{y \mid y \text{ is a lower bound of A}\}$;
7. if $\sup(A)$ exists, then $\sup(A) = \inf\{y \mid y \text{ is a upper bound of A}\}$;
8. $\sup(A)$ is unique if it exists;
9. $\inf(A)$ is unique if it exists;

Answer

1. Assume without loss of generality that $\inf(A) \leq \inf(B)$, so that $\min(\inf(A), \inf(B)) = \inf(A)$. To show that $\inf(A \cup B) = \inf(A)$, we need to show two things, that $\inf(A)$ is a lower bound for $A \cup B$ and that if t is any lower bound for $A \cup B$, then $t \leq \inf(A)$.

   If $a \in A$, then $a \geq \inf(A)$ by definition (since $\inf(A)$ is less than or equal to every element of A). Similarly, if $b \in B$, then $b \geq \inf(B)$; since $\inf(B) \geq \inf(A)$, this yields that $b \geq \inf(A)$ for all $b \in B$. Since every element c of $A \cup B$ satisfies either $c \in A$ or $c \in B$ (or both), we see that $c \geq \inf(A)$, and so $\inf(A)$ is a lower bound for $A \cup B$.

   Let t be any lower bound for $A \cup B$. Since $t \leq c$ for every $c \in A \cup B$, we also have that $t \leq c$ for every $c \in A$. In particular, t is a lower bound for A, and so by the definition of infimum, $t \leq \inf(A)$. Therefore, $\inf(A)$ is a lower bound for $A \cup B$ that is greater than or equal to any other lower bound for $A \cup B$. That is, $\inf(A \cup B) = \inf(A)$.

2. The easiest way to do this is to begin with an intermediate fact: if $A \subset B$ and if $\sup(B)$ exists, then $\sup(A)$ exists and $\sup(A) \leq \sup(B)$. The proof uses the definition of supremum: since $\sup(B)$ exists, we have that $b \leq \sup(B)$ for all $b \in B$ and that if u is an upper bound for

B, then $\sup(B) \le u$. Since $b \le \sup(B)$ for all $b \in B$ and since $A \subset B$, we have that $a \le \sup(B)$ for all $a \in A$. In particular, A is bounded above, and so $\sup(A)$ exists. To see the second statement, note that since $\sup(B)$ is an upper bound for A, we have that $\sup(A) \le \sup(B)$ by definition.

So, since $A \cap B \subset A$, we have that $\sup(A \cap B) \le \sup(A)$. Similarly, $A \cap B \subset B$, and so $\sup(A \cap B) \le \sup(B)$. Hence, $\sup(A \cap B) \le \min(\sup(A), \sup(B))$.

To have an example in which $\sup(A \cap B) < \min(\sup(A), \sup(B))$, take $A = \{0, 1\}$ and $B = \{0, 2\}$. Then, $\sup(A) = 1$, $\sup(B) = 2$, and $\sup(A \cap B) = 0$ since $A \cap B = \{0\}$.

3. The easiest way to do this is to begin with an intermediate fact: if $A \subset B$ and if $\inf(B)$ exists, then $\inf(A)$ exists and $\inf(A) \ge \inf(B)$. The proof uses the definition of infimum: since $\inf(B)$ exists, we have that $b \ge \inf(B)$ for all $b \in B$ and that if t is a lower bound for B, then $\inf(B) \ge t$. Since $b \ge \inf(B)$ for all $b \in B$ and since $A \subset B$, we have that $a \ge \inf(B)$ for all $a \in A$. In particular, A is bounded below, and so $\inf(A)$ exists. To see the second statement, note that since $\inf(B)$ is a lower bound for A, we have that $\inf(A) \ge \inf(B)$ by definition.

   So, since $A \cap B \subset A$, we have that $\inf(A \cap B) \ge \inf(A)$. Similarly, $A \cap B \subset B$, and so $\inf(A \cap B) \ge \inf(B)$. Hence, $\inf(A \cap B) \ge \max(\inf(A), \inf(B))$.

   We note that it is possible to construct an example in which $\inf(A \cap B) > \max(\inf(A), \inf(B))$. Namely, take $A = \{-1, 0\}$ and $B = \{-2, 0\}$. Then, $\inf(A) = -1$, $\inf(B) = -2$, and $\inf(A \cap B) = 0$ since $A \cap B = \{0\}$.

4. Since u is an upper bound for A, we have that $u \ge \sup(A)$, by the definition of supremum. (And note that $\sup(A)$ exists since A is bounded above.) Since $u \in A$, we also have that $u \le \sup(A)$. Since $u \ge \sup(A)$ and $u \le \sup(A)$, it must be that $u = \sup(A)$.

5. Since t is a lower bound for A, we have that $t \le \inf(A)$, by the definition of infimum. (And note that $\inf(A)$ exists since A is bounded below.) Since $t \in A$, we also have that $t \ge \inf(A)$. Since $t \le \inf(A)$ and $t \ge \inf(A)$, it must be that $t = \inf(A)$.

6. Set $X = \{y \mid y \text{ is a lower bound for A}\}$. By definition, $\inf(A) \in X$, since $\inf(A)$ is a lower bound for A. Now take any element y of X, so that y is a lower bound for A. Again by the definition of the infimum, $y \leq \inf(A)$. So, $\inf(A)$ is an upper bound for X and $\inf(A) \in X$, and so $\inf(A) = \sup(X) = \sup\{y \mid y \text{ is a lower bound for A}\}$. (Note that the assumption that $\inf(A)$ exists is equivalent to the assumption that A is bounded below, which insures that X is non-empty.)

7. Set $X = \{y \mid y \text{ is an upper bound for A}\}$. By definition, $\sup(A) \in X$, since $\sup(A)$ is an upper bound for A. Now take any element y of X, so that y is an upper bound for A. Again by the definition of the supremum, $y \geq \sup(A)$. So, $\sup(A)$ is a lower bound for X and $\sup(A) \in X$, and so $\sup(A) = \inf(X) = \inf\{y \mid y \text{ is an upper bound for A}\}$. (Note that the assumption that $\sup(A)$ exists is equivalent to the assumption that A is bounded above, which insures that X is non-empty.)

8. This one we argue by contradiction. Suppose that a set A has two suprema, and call them x_1 and x_2. Both x_1 and x_2 are upper bounds for A, by definition. Since x_1 is a supremum for A, it is less than or equal to all other upper bounds, and so $x_1 \leq x_2$. Similarly, since x_2 is a supremum for A, it is less than or equal to all other upper bounds, and so $x_2 \leq x_1$. Since $x_1 \leq x_2 \leq x_1$, it must be that $x_1 = x_2$, and so the supremum of A is unique. (Note that this exercise justifies why we call it 'the supremum' instead of 'a supremum'.)

9. This one we argue by contradiction. Suppose that a set A has two infima, and call them x_1 and x_2. Both x_1 and x_2 are lower bounds for A, by definition. Since x_1 is an infimum for A, it is greater than or equal to all other lower bounds, and so $x_1 \geq x_2$. Similarly, since x_2 is an infimum for A, it is greater than or equal to all other upper bounds, and so $x_2 \geq x_1$. Since $x_1 \geq x_2 \geq x_1$, it must be that $x_1 = x_2$, and so the infimum of A is unique. (Note that this exercise justifies why we call it 'the infimum' instead of 'an infimum'.)