Question
For each of the following, either give an example of a subset S of $\mathbf{R}$ satisfying the stated property, or prove that no such set exists.

1. S has a rational lower bound and $\inf(S)$ is irrational;
2. S has a rational lower bound and $\inf(S)$ is rational;
3. S has an irrational lower bound and $\inf(S)$ is rational;
4. S has an irrational lower bound and $\inf(S)$ is irrational;

Answer
[Note that each of these exercises has many, many possible solutions. And yes, it is a very silly question.]

1. Take $S = \{x \in \mathbf{R} \mid x > \sqrt{2}\}$, so that $\inf(S) = \sqrt{2}$, which is irrational, and S is also bounded below by 0, which is rational. (In fact, any set of real numbers that is bounded below has both infinitely many rational lower bounds and infinitely many irrational lower bounds.)
2. Take $S = (0, \infty)$, so that $\inf(S) = 0$, which is rational, and S is bounded below by -1, which is also rational.
3. Take $S = (2, 4)$, so that $\inf(S) = 2$, which is rational, and S is also bounded below by $-\pi$, which is irrational.
4. Take $S = (\sqrt{3}, \infty)$, so that $\inf(S) = \sqrt{3}$, which is irrational, and S is also bounded below by $\sqrt{2}$, which is also irrational.