Question

a) Prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a-n)^2} = \frac{\pi^2}{\sin^2 \pi a},$$

where a is not an integer.

b) State Rouche's Theorem.

Suppose that $g(z)$ is analytic inside and on the unit circle $|z| = 1$, and that $|g(z)| < 1$ for z on this circle. Show that there is a unique point z_0 inside the circle for which $g(z_0) = z_0$.

Answer

a) The function $f(z) = \dfrac{\pi \cot \pi z}{(a-z)^2}$ has simple poles at $z = 0, \pm 1 \cdots$.

At $z = n$ the residue is $\dfrac{1}{(a-n)^2}$.

The function has a pole of order 2 at $z = a$, and we have to calculate the residue.

By Taylor's theorem

$\pi \cot \pi z = \pi \cot \pi a - \pi^2 \csc^2 \pi a (z-a) +$ terms of higher degree in $(z-a)$.

So $\dfrac{\pi \cot \pi z}{(z-a)^2} = \dfrac{\pi \cot \pi a}{(z-a)^2} - \dfrac{\pi^2 \csc^2 \pi a}{(z-a)} + g(z)$ (which is analytic)

So the residue is $-\pi^2 \csc^2 \pi a$

or, we can use the formula $\operatorname{res}(f, a) = \left. \dfrac{d}{dz}(z-a)^2 f(z) \right|_{z=a}$

Now let C_N be the square with vertices $\pm(N + \frac{1}{2})(1 \pm i) \quad N \geq 0$

On the upper sides parallel to the real axis $z = x + (N + \frac{1}{2})i$.

$$|\cot \pi z| = \left| \frac{\cos \pi z}{\sin \pi z} \right| = \left| i \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| = \left| \frac{e^{2i\pi z} + 1}{e^{2i\pi z} - 1} \right|$$

$$\leq \frac{1+|e^{2\pi iz}|}{1-|e^{2\pi iz}|} = \frac{1+e^{-2\pi(N+\frac{1}{2})}}{1-e^{-2\pi(N+\frac{1}{2})}} \leq \frac{2}{1-e^{-\pi}} (N \geq 0) \text{ for all } N.$$

Also since $|\cot \pi z| = |\cot \pi(-z)|$ the same bound serves on the bottom side of the square.

On the sides parallel to the imaginary axis $z = \pm(N+\frac{1}{2}) + iy$.

$|\cot \pi z| = |\cot \pi(\pm N + \frac{1}{2} + iy)|$

$= |\cot \pi(\frac{1}{2} + iy)| = |-\tan \pi iy| = |\tanh y| \leq 1$

So $\exists K$ independent of N, such that $|\pi \cot \pi z| \leq K$ for $z \in C_N$.

Now provided $N \geq |a|$, we have

$$\int_{C_N} f(z)dz = 2\pi i \left\{ \sum_{n=-N}^{N} \frac{1}{(a-n)^2} - \pi^2 \csc^2 \pi a \right\}$$

Now $\left| \int_{C_N} \frac{\pi \cot \pi z}{(a-z)^2} dz \right| \leq \frac{K8\left(N+\frac{1}{2}\right)}{\left(N+\frac{1}{2}-|a|\right)^2} \to 0$ as $N \to \infty$.

since $|z| \geq N + \frac{1}{2}$ on C_N.

Letting $N \to \infty$ therefore gives $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(a-n)^2} = \frac{\pi^2}{\sin^2 \pi a}$

b) Rouche's Theorem is as follows:

If $f(z)$ and $g(z)$ are both analytic inside and on the closed contour C, and if $|g(z)| < |f(z)|$ on C then $f(z)$ and $F(z) + g(z)$ have the same number of zeros inside C, (counting multiplicities).

Let $f(z) = -z$. Then for z on C

$|f(z)| = |z| = 1 > |g(z)|$

SO by Rouche's theorem $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C. $f(z) = 0$ only for $z = 0$ - a simple zero, so $g(z) - z = 0$ has a unique solution z_0 inside C.

i.e. $g(z_0) = z_0$.