Question

a) Find a rational function $R(z)$ having the following properties:

   i) The only singularities are poles of order 3 at $z = +i$ and $z = -i$, and a simple pole at $z = 0$ with residue 2,

   ii) R has a zero of order 2 at $z = 1$,

   iii) $\lim_{|z|\to\infty} z^3 R(z) = 1$,

   iv) $R(-1) = -1$.

   What is the behaviour of $R(z)$ at infinity?

b) Find the Laurent series for

$$f(z) = \frac{1}{(z-a)(z-b)} \qquad (|a| < |b|)$$

   in the regions

   i) $|a| < |z| < |b|$,

   ii) $|z| > |b|$.

Answer

a) Let $R(z) = \dfrac{P(z)}{Q(z)}$

   Then by (i) $Q(z) = (z-i)^3(z+i)^3 z = z(z^2+1)^3$

   by (ii) $P(z) = (z-1)^2 M(z)$

   by (iii) $z^3 R(z)$ has a non-zero finite limit as $|z| \to \infty$, so $z^3 P(z)$ and $Q(z)$ have the same degree, namely 7. So $M(z)$ is quadratic.

   Thus $R(z) = \dfrac{(z-1)^2(Az^2+Bz+C)}{z(z^2+1)^3}$

   $\lim_{|z|\to\infty} z^3 R(z) = A$ so $A = 1$

$$\text{res}(0) = \lim_{z\to 0} zR(z) = \lim_{z\to 0} \frac{(z-1)^2(z^2+Bz+C)}{(z^2+1)^3} = C = 2$$

Finally by (iv)

$$R(-1) = \frac{(-2)^2((-1)^2 - B + 2)}{-(2)^3} = -1$$

so $3 - B = 2$ i.e. $B = 1$

Thus $R(z) = \dfrac{(z-1)^2(z^2+z+2)}{z(z^2+1)^3}$

To investigate the behaviour at infinity, consider $R(\frac{1}{z})$

$$R\left(\frac{1}{z}\right) = \frac{\left(\frac{1}{z}-1\right)^2\left(\frac{1}{z^2}+\frac{1}{z}+2\right)}{\frac{1}{z}\left(\frac{1}{z^2}+1\right)^3} = \frac{z^3(1-z)2(1+z+2z^2)}{(1+z^2)^3}$$

So $R(\frac{1}{z})$ has a zero of order 3 at $z = 0$.

i.e. $R(z)$ has a zero of order 3 at $z = \infty$.

b) $1(z-a)(z-b) = \dfrac{1}{a-b}\left(\dfrac{1}{z-a} - \dfrac{1}{z-b}\right) = f(z)$

Now for $|\alpha| > |z|$

$$\frac{1}{z-\alpha} = \frac{-1}{\alpha\left(1-\frac{z}{\alpha}\right)} = \frac{-1}{\alpha}\left(1 + \frac{z}{\alpha} + \frac{z^2}{\alpha^2} + \cdots\right)$$

whereas for $|\alpha| < |z|$

$$\frac{1}{z-\alpha} = \frac{1}{z\left(1-\frac{\alpha}{z}\right)} = \frac{1}{z}\left(1 + \frac{\alpha}{z} + \frac{\alpha^2}{z^2} + \cdots\right)$$

So (i) $|a| < |z| < |b|$

$$f(z) = \frac{1}{a-b}\left\{\frac{1}{z}\left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots\right) + \frac{1}{b}\left(1 + \frac{z}{b} + \frac{z^2}{b^2} + \cdots\right)\right\}$$

(ii) $|z| > |b|$

$$f(z) = \frac{1}{a-b}\left\{\frac{1}{z}\left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots\right) + \frac{1}{z}\left(1 + \frac{b}{z} + \frac{b^2}{z^2} + \cdots\right)\right\}$$