Question

Explain, without proofs, the relationship between differentiability of a function of a complex variable and the Cauchy-Riemann equations.

Prove that the real and imaginary parts of a differentiable function satisfy Laplace's equation.

Let $u(x, y) = \sin x \sinh y$.

Show that u is a harmonic function. Find a function v such that $f = u + iv$ is a differentiable function of $z = x + iy$.

Write down an expression for the derivative of f in terms of z.

Answer

Let $f(z) = u(x, y)+iv(x, y)$ where $z = x+iy$. The function f is differentiable at z_0 if $\dfrac{f(z_0 + h) - f(z_0)}{h}$ tends to a limit as $h \to 0$.

The function f satisfies the Cauchy-Riemann equations at z_0 if

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ at z_0.

If f is differentiable then it satisfies the Cauchy-Riemann equations. The converse is false in general, but we have a partial converse. If f satisfies the Cauchy-Riemann equations at $z_0 = x_0 + iy_0$, and if the partial derivatives exist in a neighbourhood of (x_0, y_0) and are continuous at (x_0, y_0) then f is differentiable at z_0.

Now if f is differentiable then f is analytic, so partial derivatives of all orders exist and are continuous

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Mixed partial derivatives are equal, so adding gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Also

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

subtracting and equating mixed derivatives gives

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Now $u = \sin x \sinh y$

$\dfrac{\partial u}{\partial x} = \cos x \sinh y \qquad \dfrac{\partial u}{\partial y} = \sin x \cosh y$

$\dfrac{\partial^2 u}{\partial x^2} = -\sin x \sinh y \qquad \dfrac{\partial^2 v}{\partial y^2} = \sin x \sinh y$

So $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ i.e. u is harmonic.

$\dfrac{\partial u}{\partial x} = \cos x \sinh y = \dfrac{\partial v}{\partial y}$

so $v = \cos x \cosh y + \phi(x)$

$-\frac{\partial u}{\partial y} = -\sin x \cosh y = \frac{\partial v}{\partial x}$

so $v = \cos x \cosh y + \psi(y)$

so $\phi(x) = \psi(y)$=constant

Thus $v = \cos x \cosh y + c$

$$\begin{aligned} f &= \sin x \sinh y + i \cos x \cosh y \\ &= \sin x(-i \sin iy) + i \cos x \cos iy \\ &= i(\cos x \cos iy - \sin x \sin iy) \\ &= i \cos z \end{aligned}$$

So $\dfrac{df}{dz} = -i \sin z$

Or $\dfrac{df}{dz} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$= \cos x \sinh y - i \sin x \cosh y$

$= -i(\sin x \cos iy + \cos x \sin iy)$

$= -i \sin z$