QUESTION

Spherical polar coordinates (r, ϕ, θ) are defined in terms of cartesian coordinates (x, y, z) by the equations

$$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi,\ \ z = r\cos\theta.$$

Calculate r, θ and ϕ in terms of x, y and z, and then the nine partial derivatives $\frac{\partial r}{\partial x}$ etc.

ANSWER

$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi, z = r\cos\theta$
$r = (x^2+y^2+z^2)^{\frac{1}{2}},\ \tan\theta = (x^2+y^2)^{\frac{1}{2}}z^{-1},\ \tan\phi = yx^{-1}$

$$\frac{\partial r}{\partial x} = 2x\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$\dfrac{d\arctan u}{du} = \dfrac{1}{1+u^2}$

$\dfrac{\partial\theta}{\partial z} = \dfrac{\partial\arctan(x^2+y^2)^{\frac{1}{2}}z^{-1}}{\partial z} = \dfrac{1}{1+(x^2+y^2)z^{-2}}(x^2+y^2)^{\frac{1}{2}}(-z^{-2})$

$= \dfrac{-(x^2+y^2)^{\frac{1}{2}}}{r^2},$

$\dfrac{\partial\theta}{\partial x} = \dfrac{1}{1+(x^2+y^2)z^{-2}}2x\dfrac{1}{2}(x^2+y^2)^{-\frac{1}{2}}z^{-1} = \dfrac{xz}{(x^2+y^2)^{\frac{1}{2}}r^2},$

$\dfrac{\partial\theta}{\partial y} = \dfrac{yz}{(x^2+y^2)^{\frac{1}{2}}r^2}$

$\dfrac{\partial\phi}{\partial x} = \dfrac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\dfrac{y}{x^2}\right) = -\dfrac{y}{x^2+y^2},\ \dfrac{\partial\phi}{\partial y} = \dfrac{1}{1+\left(\frac{y}{x}\right)^2}\dfrac{1}{x} = \dfrac{x}{x^2+y^2},\ \dfrac{\partial\phi}{\partial z} = 0$