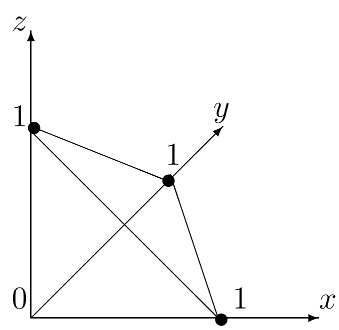
QUESTION

(a) Evaluate $\iint \mathbf{A} \cdot \mathbf{dS}$ for each of the three right-angled triangular faces of the prism bounded by $x = 0, y = 0, z = 0$ and $x + y + z = 1$ given that $\mathbf{A} = (2z, -y, 3x)$.

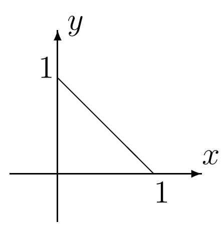
(b) By using (a) and the divergence theorem, evaluate the surface integral over the remaining face.

ANSWER

(a)



$\mathbf{A} = (2z, -y, 3x), \quad z = 0$ plane, $\mathbf{n} = (0, 0, -1)$

and the triangle 

is the surface, with $z = 0$.

$$\begin{aligned}
I_z &= \int_0^1 dx \int_0^{1-x} dy(0, -y, 3x) \cdot (0.0. - 1) \\
&= \int_0^1 dx \int_0^{1-x} dy(-3x) \\
&= \int_0^1 (-3x)(1-x)\, dx \\
&= \int_0^1 (3x^2 - 3x)\, dx \\
&= \left[x^3 - \frac{3}{2}x^2\right]_0^1 = -\frac{1}{2}.
\end{aligned}$$

Similarly

$$
\begin{aligned}
I_y &= \int_0^1 dz \int_0^{1-z} dz(2z,0,3x)(0,-1,0) = 0 \\
\text{and} & \\
I_x &= \int_0^1 dy \int_0^{1-y} dz(2z,-y,0)(-1,0,0) \\
&= \int_0^1 dy \int_0^{1-y} (-2z)\, dz \\
&= \int_0^1 dy \left[-z^2\right]_0^{1-y} \\
&= dy(-(1-y)^2) \\
&= \int_0^1 dy(-y^2+2y-1) = \left[-\frac{y^3}{3}+y^2-y\right]_0^1 = -\frac{1}{3}
\end{aligned}
$$

(b) $\nabla \cdot \mathbf{A} = -1$

$\iint_S \mathbf{A} \cdot \mathbf{dS} = \iiint_V \nabla \cdot \mathbf{A} dV = -1$. Volume $= -\frac{1}{6}$ because $\nabla \cdot \mathbf{A}$ is constant.

But $\iint_s \mathbf{A} \cdot \mathbf{dS} = -\frac{1}{2} \cdot -\frac{1}{3} + I_4 = -1 \Rightarrow I_4 = -\frac{1}{6}$ where I_4 is the flux of A through the fourth side of the prism.