Question

Let γ be the space curve given by

$$\gamma(t) = (\cosh t, \sinh t, 2t).$$

Show that γ has a vertex ($\dot{\kappa} = 0$) at $P = (1,0,0)$ and at points A, B given by solutions t to $\cosh 2t = 13/8$. Calculate the torsion at P, A and B.

Answer

$$\begin{aligned}
\gamma(t) &= (\cosh t, \sinh t, 2t) \\
\dot{\gamma}(t) &= (\sinh t, \cosh t, 2) \\
\dot{\gamma} \cap \ddot{\gamma} &= (-2\sinh t, 2\cosh t, -1) \\
\dot{\gamma} \cap \ddot{\gamma}.\ \dddot{\gamma} &= -2\sinh^2 t + 2\cosh^2 t = 2
\end{aligned}$$

Hence

$$K^2 = \|\dot{\gamma} \cap \ddot{\gamma}\|^2 \|\dot{\gamma}\|^{-6} = (1 + 4\cosh 2t)(4 + \cosh 2t)^{-3}$$

Note that $K \neq 0$.

Differentiate:

$$2K\dot{K} = \cdots = 2(t + \cosh 2t)^{-4}[4(4 + \cosh 2t) - 3(1 + 4\cosh 2t)]\sinh 2t$$

This is $= 0$ **when** $\sinh 2t = 0$ **or** $13 - 8\cosh 2t = 0$**.**

Now, $\tau = (\dot{\gamma} \cap \ddot{\gamma}.\ \dddot{\gamma})\|\dot{\gamma} \cap \ddot{\gamma}\|^{-2} = 2(1 + 4\cosh 2t)^{-1}$**.**

Hence at P **(**$t = 0$**) we have** $\tau = \frac{2}{5}$

while at A**,**B **(**$\cosh 2t = \frac{13}{8}$**) we have** $\tau = 2(1 + \frac{13}{2})^{-1} = \frac{4}{15}$**.**