Question

(i) Show that the space curve $\gamma(t) = (t, t^{-1}(1+t), t^{-1}(1-t^2))$ is in fact planar.
(ii) Find a differential equation that the function $g(t)$ must satisfy in order for the curve

$$\gamma(t) = (\cos t, \sin t, g(t))$$

to be planar. Hence find the most general such function $g(t)$.
[*Hint: $B(t)$ has constant direction.*]
Answer

(i) Observe that $\gamma(t) = \left(t, \frac{1}{t}+1, \frac{1}{t}-1\right)$ which lies in $x - y + z = -1$,

OR: Check that $\dot\gamma \cap \ddot\gamma = \left(\frac{2}{t^3}, -\frac{2}{t^3}, \frac{2}{t^3}\right) = \frac{2}{t^3}(1,-1,1)$

so $B(t) = \dot\gamma \cap \ddot\gamma / \|\dot\gamma \cap \ddot\gamma\| = \frac{1}{\sqrt{3}}(1,-1,1)$, i.e. constant.

(ii) Here $\dot\gamma \cap \ddot\gamma = (\cos t\ddot g + \sin t\dot g, \sin t\ddot g - \cos t\dot g, 1)$; then γ is planar precisely when this vector has constant direction, $= (a, b, 1)$ say.

Solving for $(\dot g, \ddot g)$ we find $\dot g = a\sin t - b\cos t$ (and $\ddot g = a\cos t + b\sin t$).

So $g(t) = A\cos t + B\sin t + C$ for arbitrary constants A, B and C.

OR: $I = 0$ when $\dot\gamma \cap \ddot\gamma . \dddot\gamma = 0$ which leads to $\dddot g + \dot g = 0$. Hence $\dot g = h$ satisfies $\ddot h + h =$), so $h = \alpha\cos t + \beta\sin t$.

$$\Rightarrow g(t) = A\cos t + B\sin t + C.$$