Question

Find the evolute of each curve γ, both as (i) the locus of centres of curvature and (ii) the envelope of the normals:

(a) $\gamma(t) = (t - \sin t, 1 - \cos t)$

(b) $\gamma(t) = (2\cos t + \cos 2t, 2\sin t - \sin 2t)$.

Answer

(a)

$$\begin{aligned}
\gamma(t) &= (t - \sin t, 1 - \cos t) \\
\dot{\gamma}(t) &= (1 - \cos t, \sin t) \\
\ddot{\gamma}(t) &= (\sin t, \cos t) \\
K(t) &= -1/4 \sin \frac{t}{2}
\end{aligned}$$

Now

$$\begin{aligned}
N(t) &= (-\sin t, 1 - \cos t)/(\sin^2 t + (1 \cos t)^2)^{\frac{1}{2}} \\
&= (2 - 2\cos t)^{-\frac{1}{2}}(-\sin t, 1 - \cos t) \\
&= \frac{1}{2\sin\frac{t}{2}}(-\sin t, 1 - \cos t)
\end{aligned}$$

Hence centre of curvature is

$$\begin{aligned}
\gamma(t) + (K(t))^{-1}N(t) &= (t - \sin t, 1 - \sin t) \\
&= 2(-\sin t, 1 - \cos t) \\
&= (t + \sin t, -1 + \cos t)
\end{aligned}$$

OR: Equation of normal at $\gamma(t)$ is

$$x(1-c) + ys = (t-s)(1-c) + (1-c)S = t(1-c)$$

Differentiation with respect to t gives

$$xs + yc = 1 - c + ts$$

Solving for x, y gives

$$(x, y) = (t + \sin t, -1 + \cos t)$$

(Observe that the evolute is also a cycloid: put $t = \pi + u$, and translate by $(\pi, -2)$.)

(b)

$$\begin{aligned}
\gamma(t) &= (2\cos t + \cos 2t, 2\sin t - \sin 2t), \quad \textbf{hypocycloid} \\
\dot\gamma(t) &= (-2\sin t - 2\sin 2t, 2\cos t - 2\cos 2t) \\
\ddot\gamma(t) &= 2(-\cos t - 2\cos 2t, -\sin t + 2\sin 2t)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{4}\|\dot\gamma\|^2 &= (\sin t + \sin 2t)^2 + (\cos t - \cos 2t)^2 \\
&= 2 + 2(\sin t \sin 2t - \cos t \cos 2t) \\
&= 2 - 2\cos 3t = 4\sin^2 \frac{3t}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}(\dot x \ddot y - \dot y \ddot x) &= (-\sin t - \sin 2t)(-\sin t + 2\sin 2t) \\
&\quad +(\cos t - \cos 2t)(\cos t + 2\cos 2t) \\
&= 1 - 2 - \sin t \sin 2t + \cos t \cos 2t \\
&= -(1 - \cos 3t) = -2\sin^2 \frac{3t}{2} \\
\Rightarrow K(t) &= -4\sin^2 \frac{3t}{2} \Big/ \left(4\sin\frac{3t}{2}\right)^3 \\
&= -1/16 \sin \frac{3t}{2}
\end{aligned}$$

Now

$$N(t) = \frac{1}{4\sin\frac{3t}{2}}(-\cos t + \cos 2t, -\sin t - \sin 2t),$$

so centre of curvature is

$$\begin{aligned}
\gamma(t) + (K(t))^{-1}N(t) &= (2\cos t + \cos 2t, 2\sin t - \sin 2t) \\
&\quad -4(-\cos t + \cos 2t, -\sin t - \sin 2t) \\
&= 3(2\cos t - \cos 2t, 2\sin t + \sin 2t)
\end{aligned}$$

Which we see is another hypocycloid, $= -3 \times \gamma(t + \pi)$**.**

OR: Equation of normal at $\gamma(t)$ **is**

$$-x(\sin t + \sin 2t) + y(\cos t - \cos 2t) = \ \cdots \ = -3\sin 3t$$

Differentiate with respect to t **to give**

$$-x(\cos t + 2\cos 2t) + y(-\sin t + 2\sin 2t) = -9\cos 3t$$

'Solving' for x, y **gives (eventually)**

$$(x, y) = 3(2\cos t - \cos 2t, 2\sin t + \sin 2t)$$

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