Question
Use the method of matching to find the first terms in the outer and inner solutions of

$$\varepsilon y'' + y' + y = 0,\ y(0) = 1,\ y(1) = 1,$$

given that a boundary layer of width $O(\varepsilon)$ exists near the origin. Hence write down the one-term composite expansion. Compare thus with the exact solution.

Answer
$\varepsilon y'' + y' + y = 0,\ y(0) = 1,\ y(1) = 1$
Boundary layer of $O(\varepsilon)$ exists at origin
<u>OUTER</u> $y = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$
Substitute into equations:

$$\varepsilon y_0 + (\varepsilon^2 y_1) + y_0' + \varepsilon y_1' + y_0 + \varepsilon y_1' = O(\varepsilon^2)$$

Balance at
$\underline{O(\varepsilon^0)} : +y_0' + y_0 = 0 \Rightarrow y_0 = Ae^{-x}$
Boundary condition: $y_0(0) = 1$ (Note in outer region. Therefore irrelevant), $y_0(1) = 1$ (Only this one is relevant)
$\Rightarrow \underline{y_0 = e^{-x+1}}$
<u>INNER</u>
Given boundary layers is $O(\varepsilon)$.
Use inner variable $X = \dfrac{x}{\varepsilon}$: $\partial_x = \dfrac{\partial X}{\partial x}\partial_X = \dfrac{1}{\varepsilon}\partial_x$ and set $y(\varepsilon x; \varepsilon) = Y(X; \varepsilon)$
Equation becomes:
$\dfrac{1}{\varepsilon}Y;; + \dfrac{1}{\varepsilon}Y' + Y = 0,\ Y(0) = 1$
$\Rightarrow Y'' + Y' + \varepsilon Y = 0,\ Y(0) = 1$
2nd order equation and only one boundary condition $\Rightarrow$ matching is needed.
Try regular ansatz: $Y(X; \varepsilon) = Y_0(X) + \varepsilon Y_1(X) + \varepsilon^2 Y_2(X) + O(\varepsilon^3)$
$\Rightarrow Y_0'' + \varepsilon Y_1'' + Y_0' - \varepsilon Y_1' + \varepsilon Y_0 + (\varepsilon^2 Y_1) = O(\varepsilon^2)$

$$\begin{aligned} & & Y_0'' + Y_0' = 0; Y_0(0) = 1 \\ \underline{O(\varepsilon^0)}: & \Rightarrow & Y_o' + Y_0 = const = A \text{ say} \\ & \Rightarrow & Y_0 = +A + Ce^{-x} \text{ where } A \text{ and } C \text{ are arbitary constants} \end{aligned}$$

Use <u>1</u> boundary condition $Y_0(0) = 1 \Rightarrow 1 = A + C$,
only one constant can be determined, say $A = 1 - C$
Therefore $Y_0 = (1 - C) + Ce^{-X}$
Match up to get value of C using Van Dyke.

$$
\begin{array}{lcl}
\text{One term outer expansion} & = & e^{1-x} \\
\text{Rewritten in inner variable} & = & e^{1-\varepsilon X} \\
\text{Expanded for small } \varepsilon & = & e(1-\varepsilon x+O(\varepsilon^2)) \\
\text{One term } O(\varepsilon^0) & = & e+O(\varepsilon)\ (\star) \\
\text{One term inner expansion} & = & -(c-1)+ce^{-X} \\
\text{Rewritten in outer variable} & = & -(c-1)+ce^{-\frac{X}{\varepsilon}} \\
\text{Expanded for small } \varepsilon & = & -(c-1) \\
 & & +\text{exp. small term in}\varepsilon \\
 & & (\text{no +ve power series in } t) \\
\text{One term } O(\varepsilon^0) & = & -c+1\ (\star\star)
\end{array}
$$

$(\star) and (\star\star)$ must be equal:

$\Rightarrow e=-c+1$ or $c=1-e$

Therefore outer 1-term is: $y(x;\varepsilon)=e^{1-x}+O(\varepsilon)$

Inner 1-term is $Y(X;\varepsilon)=e+(1-e)e^{-X}$

Composite is:

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\begin{aligned}
y^{comp} &= y^{outer}+y^{inner}-\text{outer limit of } y^{inner} \text{ or inner limit of } y^{outer} \\
&= e^{1-x}+e+(1-e)e^{-X}-e \quad (+O(\varepsilon)) \\
&= e^{1-x}+(1-e)e^{-X}+\cdots \\
&= e^{1-x}+(1-e)e^{-\frac{x}{\varepsilon}}+\cdots,\ \varepsilon\to o^+
\end{aligned}
$$

Exact solution is given by substituting $y=Ae^{kx}$

$\Rightarrow \varepsilon k^2+k+1=0 \Rightarrow k=\dfrac{-1\pm\sqrt{1-4\varepsilon}}{2\varepsilon}$

General solution is $y=Ae^{\overbrace{\left(\frac{-1+\sqrt{1-4\varepsilon}}{2\varepsilon}\right)}_{k_1}x}+Be^{\overbrace{\left(\frac{-1-\sqrt{1-4\varepsilon}}{2\varepsilon}\right)}_{k_2}x}$

$y(0)=1\Rightarrow A+B=1$

$y(1)=1\Rightarrow Ae^{k_1}+Be^{k_2}=1$

so $A=\left(\dfrac{e^{k_2}-1}{e^{k_2}-e^{k_1}}\right),\ B=\left(\dfrac{1-e^{k_1}}{e^{k_2}-e^{k_1}}\right)$ so exact solution is

$$
y=\frac{(e^{k_2}-1)e^{k_1x}+(1-e^{k_1})e^{k_2x}}{e^{k_2}-e^{k_1}}
$$

Small ε expansion gives $k_1=-1+O(\varepsilon),\ k_2=-\dfrac{1}{\varepsilon}+1+O(\varepsilon).e^{k_2}$ is therefore exp. small.

$$
\Rightarrow y\sim +e^{k_1(x-1)}+(1-e^{-k_1})e^{k_2x}\sim e^{1-x}+(1-e)e^{-\frac{x}{\varepsilon}}\ \surd\surd
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