Question
Compute the coefficients in the perturbation series solution of the equation

$$y' = y + \varepsilon xy,\ \ y(0) = 1$$

Answer
$y' = y + \varepsilon xy,\ \ y(0) = 1$
Try $y(x;\varepsilon) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2) = \sum_{n=0}^{\infty} y_n \varepsilon^n$
Substitute into equation:

$$\sum_{n=0}^{\infty}(yn'\varepsilon^n - y_n\varepsilon^n) = \varepsilon \sum_{n=0}^{\infty} xy_n\varepsilon^n$$

Balance at $O(\varepsilon^n)$:
$\underline{n=0}: y_0' - y_0 = 0 \Rightarrow y_0 = Ae^x$
Boundary condition: $y_0(0) = 1 \Rightarrow 1 = Ae^0 \Rightarrow A = 1 \Rightarrow \underline{y_0 = e^x}$
$n > 0 : y_n' - y_n = xy_{n-1}$
e.g.,

$$\underline{n=1}: \begin{array}{rcl} y_1' - y_1 &=& xe^x \\ \Rightarrow \frac{d}{dx}\left(e^{-x}y_1\right) &=& x \\ \Rightarrow e^{-x}y_1 &=& \frac{x^2}{2} + c \\ y_1 &=& \frac{x^2}{2}e^x + ce^x \end{array}$$

Boundary condition: $y_1(0) = 0$
$0 = 0 + ce^x \Rightarrow c = 0$
Therefore $\underline{y_1 = \frac{x^2}{2}e^x}$

$$\text{then } \underline{n=2}: \begin{array}{rcl} y_2' - y_2 &=& \frac{x^3}{2}e^x \\ \Rightarrow \frac{d}{dx}\left(e^{-x}y_2\right) &=& \frac{x^3}{2} \\ \Rightarrow e^{-x}y_2 &=& \frac{x^4}{8} + c \\ y_2 &=& \frac{x^4}{8}e^x + ce^x \end{array}$$

Boundary condition: $y_2 90) = 0 \Rightarrow c = 0$
Therefore $y_2 = \frac{x^4}{8}e^x$
Spot the pattern.
In general:

$$
\begin{aligned}
y_n' - y_n &= \frac{x^{2n-1}e^x}{2^{n-1}(n-1)!} \quad n \geq 1 \\
& y_n(0) = 0 \\
\Rightarrow y_n &= \frac{x^{2n}e^x}{2^n n!} \\
\text{Therefore y} &= \sum_{n=0}^{\infty} \frac{\varepsilon^n x^{2n}}{2^n n!} e^x \quad (\star)
\end{aligned}
$$

Compare with <u>exact</u> solution

$\frac{d}{dx}\left(e^{-\int (1+\varepsilon x)\,dx} y\right) = 0 \Rightarrow y = ce^{x+\frac{\varepsilon x^2}{2}}$

$y(0) = 1 \Rightarrow c = 1$

Therefore $y = e^{x+\frac{\varepsilon x^2}{2}} = e^x \sum_{n=0}^{\infty} \frac{\varepsilon^n x^{2n}}{2^n n!} = (\star)$ $\surd\surd$ (Convergent for all finite x)