Question
Consider van der Pol's equation

$$u'' + u = \varepsilon(1 - u^2)u', \ \varepsilon \to 0^+$$
$$u(0) = 2, \ u'(0) = 0$$

Show that substitution of the regular ansatz

$$u(x;\varepsilon) = u_0(x) + \varepsilon u_1(x) + O(\varepsilon^2)$$

leads to a perturbation solution for finite x,

$$u(x;\varepsilon) = 2\cos x + \varepsilon\left(\frac{3}{4}\sin x - \frac{\sin 3x}{2}\right) + O(\varepsilon^2)$$

Answer
$u'' + u = \varepsilon(1 - u^2)e' \ \ \varepsilon \to 0^+ \ \ u(0) = 2, \ u'(0) = 0.$
Substitute $u(x;\varepsilon) = u_0(x) + \varepsilon u_1(x) + O(\varepsilon^3)$????
$u_0'' + \varepsilon u_1{}'' + u_0 + \varepsilon u_1 + O(\varepsilon^2) = \varepsilon(1 - u_0^2 - \varepsilon^2 u_1 - 2u_0 u_1 \varepsilon + O(\varepsilon^2)) \times (u_0' + \varepsilon u_1' + O(\varepsilon^2))$
Collect terms of $O(\varepsilon^0)$ and $O(\varepsilon^1)$
$O(\varepsilon^0) : u_0{}'' + u_0 = 0 \Rightarrow u_0 = A\sin x + B\cos x$
Boundary conditions: $u_0(0) = 2, \ u_0'(0) = 0$
$\Rightarrow A = 0, \ B = 2 \Rightarrow u_0 = 2\cos x$
$O(\varepsilon) : u_1;; + u_1 = u_0'(1 - u_0^2)$
$u_0 = 2\cos x, \ u_0' = -2\sin x \Rightarrow RHS = -2 sinx(1 - 4\cos^2 x)$

$$\text{But } 4\sin^3 x - 3\sin x = -\sin 3x \begin{cases} = -2\sin x(1 - 4 + 4\sin^2 x) \\ = -2\sin x(-3 + 4\sin^2 x) \\ = -2(4\sin^3 x - 3\sin x) \end{cases}$$

Therefore $u_1{}'' + u_1 = +2\sin 3x \ \ (A)$

$$u_1 = u_1^{CF} + u_1^{PI} \rightarrow \begin{array}{l} u_1^{CF} : u_1^{CF''} + u_1^{CF} = 0 \\ u_1^{PI} : u_1^{PI} = C\sin 3x + D\cos 3x \end{array}$$

By substituting in (A):
$-9C\sin 3x - 9D\cos 3x + C\sin 3x + D\cos 3x = 2\sin 3x$
$\Rightarrow -8C\sin 3x - 8D\cos 3x = 2\sin 3x$
$\Rightarrow C = -\frac{1}{4}, \ D = 0$
and $u_1^{CF} = E\sin x + D\cos x$
Therefore $u_1 = E\sin x + D\cos x - \frac{1}{4}\sin 3x$

Boundary conditions:
$u_1(0) = u_1'(0) = 0$ (from perturbation of boundary conditions) $\Rightarrow D = 0$

$u_1'(0) = E\cos 0 - \frac{3}{4}\cos 0 \Rightarrow E = \frac{3}{4}$

Therefore $u(x;\varepsilon) \sim 2\cos x + \varepsilon\left(\frac{3}{4}\sin x - \frac{\sin 3x}{2}\right)$