Question
(More difficult) Consider the problem

$$\varepsilon y'' = y^2 = 0,\ y(0) = 1,\ y(1) = 1,\ \varepsilon \to 0^+.$$

Try to solve this when $0 \le x \le 1$ by using a regular perturbation scheme of the type

$$y(x;\varepsilon) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$$

Show that it is not possible for thus outer type of expansion to satisfy the boundary conditions at either end. Hence deduce that there must be a boundary layer near both $x = 0$ and $x = 1$.
You are given that both boundary layers have a width of order $\varepsilon^{\frac{1}{2}}$. Determine one term of the inner solution near to the origin in the standard way. Repeat this for the second inner solution near to $x = 1$. match the two inner expansions to the common outer expansion in the relevant regions.
Hint: The solution $Y'' - Y^2 = 0$ can be obtained by first reducing the order with the substitution $u = Y'$ (hence $Y'' = u' = \frac{du}{dY}\frac{dY}{dx} = u\frac{du}{dY}$), solving for u, and then resubstituting for Y in the resulting equation. Also use the fact that if $\lim_{x\to\infty} Y = 0$ then $\lim_{x\to\infty} Y' = 0$.

Answer
This is difficult: it has 2 boundary layers and $O(\varepsilon^{\frac{1}{2}})$ width.
Given

$$\varepsilon y'' - y^2 = 0;\ y(0) = 1,\ y(1) = 1;\ \varepsilon \to 0^+$$

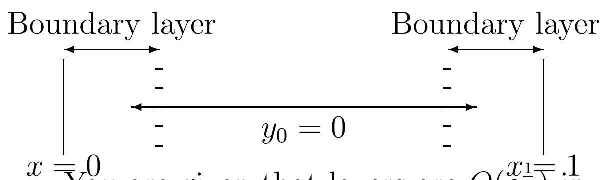
we try the usual ansatz anyway:

$$y(x;\varepsilon) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$$

$\Rightarrow \varepsilon y_0'' - y_0^2 - 2\varepsilon y_0 y_1 = O(\varepsilon^2);\ y_0(1) = 1,\ y_r(1) = 0\ (r > 0)$

$$\begin{aligned} -y_0^2 &= 0 \\ \underline{O(\varepsilon^0)} \Rightarrow y - 0 &= 0 \text{ but } \mathrm{y}_0(1) = 1 \text{ from boundary data.} \end{aligned}$$

Therefore inconsistency.
Hence usual ansatz <u>can't work</u> at $x = 0$.
Equally, if we think of using ansatz at $x = 0$ we run into the same problem: $y_0 = 0$ by $y_0(0) = 1$.
Hence we must have a boundary layer at both ends:



You are given that layers are $O(\varepsilon^{\frac{1}{2}})$ in width.

<u>Near $x = 0$</u>

Try inner variable $X = \dfrac{x}{\varepsilon^{\frac{1}{2}}}$

$$\Rightarrow \partial_x = \frac{1}{\varepsilon^{\frac{1}{2}}}\partial_X$$

and use ansatz $y(\varepsilon^{\frac{1}{2}}X;\varepsilon) = Y(X;\varepsilon)$

$$Y(X;\varepsilon) = Y_0(X) + \varepsilon^{\frac{1}{2}}Y_1(X) + O(\varepsilon)$$

Substitute into equation:

$$\frac{\varepsilon}{\varepsilon}Y_0'' - \frac{\varepsilon}{\varepsilon}\varepsilon^{\frac{1}{2}}Y_1''(X) - Y_0^2 - 2\varepsilon^{\frac{1}{2}}Y_0Y_1 = 0$$

$$Y_0'' - Y_0^2 = O(\varepsilon^{\frac{1}{2}})$$

Boundary date: only $x = 0$ is relevant and we get $y(0) = 1 \Rightarrow Y(0) = 1 \Rightarrow$ $Y_0(0) = 1,\ Y_{r>0}(0) = 0$

Therefore $\overline{O(\varepsilon^0)}$

$Y_0'' - Y_0^2 = 0;\ Y_0(0) = 1$

Again 2nd order equation. 1 boundary condition $\Rightarrow$ matching.

Solve this by setting $\left.\begin{array}{rcl} u & = & Y_0' \\ u' & = & Y_0'' \end{array}\right\}$ by hint of question

$u = \dfrac{dY_0}{dX} \Rightarrow \dfrac{d^2Y_0}{dX^2} = \dfrac{du}{dX} = \dfrac{du}{dY_0} \times \dfrac{dY_0}{dX} = \dfrac{du}{dY_0}Y_0' = u\dfrac{du}{dY_0}$

$$
\begin{aligned}
u\frac{du}{dY_0} - Y_0^2 &= 0 \\
\Rightarrow \int u\,du &= \int Y_0^2\,dY_0 \\
\text{Therefore} \qquad \frac{u^2}{2} &= \frac{Y_0^3}{3} + const \\
\text{or } \frac{1}{2}\left(\frac{\mathrm{dY_0}}{\mathrm{dX}}\right)^2 &= \frac{Y_0^3}{3} + c \\
\text{or } \frac{\mathrm{dY_0}}{\mathrm{dX}} &= \pm\sqrt{\frac{2}{3}Y_0^3 + 2c} \quad (\star)
\end{aligned}
$$

This is tricky to solve unless we start imposing boundary conditions. We know $Y_0(0) = 1$ but this doesn't tell us about $\frac{dY_0}{dX}$. Clearly though if things are to match up, we must have at leading order.

$$Y_0 \longrightarrow y_0 = 0 \text{ as } \varepsilon \longrightarrow 0^+$$

i.e., as $X \longrightarrow +\infty$ in the outer (middle) region
Therefore $lim_{X\to\infty} Y_0 = 0$.
Thus if $\lim_{X\to+\infty} Y_0 = 0$ the curve $Y_0(X)$ must <u>flatten</u>.
So $Y_0' \longrightarrow 0$ as $X \longrightarrow +\infty$
Therefore $0 = \pm\sqrt{0 + 2c}$ as $X \longrightarrow +\infty$ from $(\star)$.
Hence $\frac{dY_0}{dX} = \pm\sqrt{\frac{2}{3}}Y_0^{\frac{3}{2}}$
which is variables separable

$$
\begin{aligned}
\int Y_0^{-\frac{3}{2}}\,dY_0 &= \pm\int\sqrt{\frac{2}{3}}\,dX \\
-2Y_0^{-\frac{1}{2}} + D &= \pm\sqrt{\frac{2}{3}}X
\end{aligned}
$$

Now we can use the actual boundary condition: $Y_0(0) = 1$

$$
\begin{aligned}
&0 = D - 2 \Rightarrow D = 2 \\
&\text{Therefore } 2 - \frac{2}{\sqrt{\mathrm{Y_0}}} = \pm\sqrt{\frac{2}{3}}\mathrm{X} \\
&\Rightarrow Y_0 = \frac{1}{(1 \mp \sqrt{\frac{1}{6}}X)^2}
\end{aligned}
$$

Now, if we don't want a singularity for finite X, the - sign must be discarded.
Hence

$$\underline{Y_0 = \frac{1}{(1+\sqrt{\frac{1}{6}}X)^2}}$$

<u>Near $x = 1$</u>:
To determine the inner expansion.
Near $x = 1$ we use
layer near $x = 1$

$z = \frac{\overbrace{1-x}}{\underbrace{\varepsilon^{\frac{1}{2}}}}$, say

since $O(\varepsilon^{\frac{1}{2}})$ width is given to you

Therefore $\partial_x = \frac{\partial z}{\partial x}\partial_z = -\frac{1}{\varepsilon^{\frac{1}{2}}}\partial_z$
and $y(1-\varepsilon^{\frac{1}{2}}z;\varepsilon) = \bar{Y}(z;\varepsilon)$
and $\bar{Y}(z;\varepsilon) = \bar{Y}_0(z) + \varepsilon^{\frac{1}{2}}\bar{Y}_1(z) + O(\varepsilon)$
Equation becomes:

$\frac{\varepsilon}{(-\varepsilon^{\frac{1}{2}})}\frac{\partial^2\bar{Y}}{\partial z^2} - \bar{Y}^2 = 0 \Rightarrow \frac{\partial^2\bar{Y}}{\partial z^2} - \bar{Y}^2 = 0$

so $\bar{Y_0}^2 - \bar{Y_0}^2 = O(\varepsilon^{\frac{1}{2}})$
This is the same equation as at $x = 0$ so we expect a solution

$$\frac{d\bar{Y}_0}{dz} = \pm\sqrt{\frac{2}{3}\bar{Y_0}^3 + \bar{c}}$$

Again since $y(1) = 1$ $(x = 1 \Rightarrow z = 0) \Rightarrow \bar{Y}_0(0) = 1$
and since $\bar{Y}_0$ must match onto $y_0 = 0$ <u>as</u> $\varepsilon \to +\infty$
i.e., as $z \to +\infty$
Therefore we also have $\bar{Y_0}' \to 0$ as $z \to +\infty$ as before. Hence we get an identical solution:

$$\underline{\bar{Y}_0 = \frac{1}{(1+\sqrt{\frac{1}{6}}z)^2}}$$

(negative sign discarded to avoid singularity at finite z)
Rewrite in original variables and summarise:

$$\begin{cases} y \sim \frac{1}{(1+\frac{1}{\sqrt{6}}\frac{x}{\varepsilon^{\frac{1}{2}}})}, \ x = o(\varepsilon^{\frac{1}{2}}) \\ y \sim 0, \ o(\varepsilon^{\frac{1}{2}}) < x < O(\varepsilon^{\frac{1}{2}}) \\ y \sim \frac{1}{(1+\frac{1}{\sqrt{6}}\frac{(1-x)}{\varepsilon^{\frac{1}{2}}})^2}, \ x - 1 = O(\varepsilon^{\frac{1}{2}}) \end{cases}$$

Solution
PICTURE