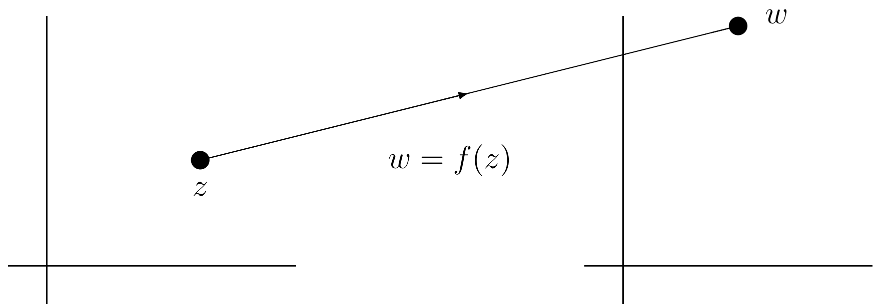
Complex Numbers

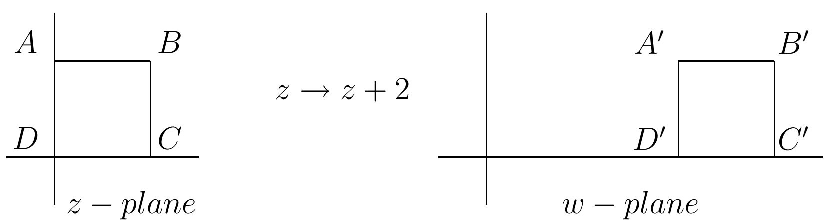
Geometrical Transformations in the Complex Plane

For functions of a real variable such as $f(x) = \sin x$, $g(x) = x^2+2$ etc you are used to illustrating these geometrically, usually on a cartesian graph. If we have functions of a complex variable given by equations such as $w = \sin z$ or $w = z^2+2$ we cannot use a cartesian graph, since z cannot be represented on an ordered axis. Indeed z may range over the whole of the two dimensional complex plane, so that if w is also complex we would need a 4-dimensional space to plot a graph such as $w = z^2 + 2$. Most of us cannot visualise this, and what we usually do is to have two copies of the complex plane, and we look at points in the z-plane and see how they are transformed into points in the w-plane. We also look at sets of points, curves or regions in the z-plane and their images in the w-plane.

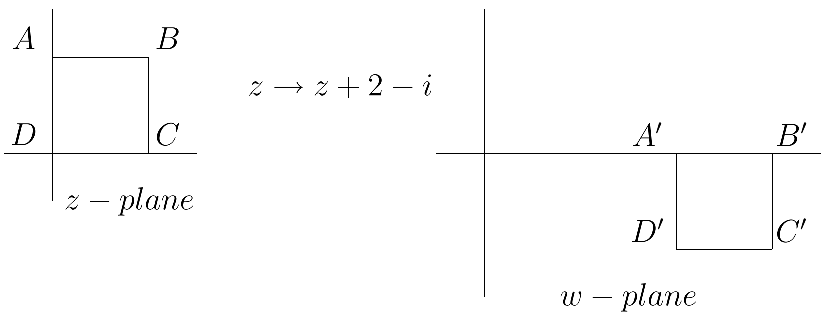


Examples

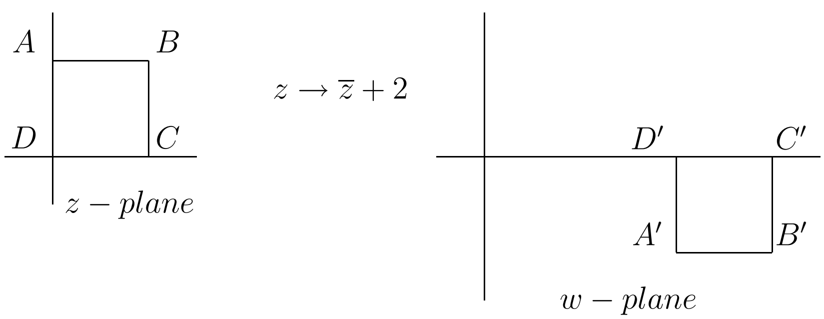
1) $w = f(z) = z + 2$. This simply shifts every point two units in the direction of the real axis - it is a translation.



2) $w = z + 2 - i$, again a translation

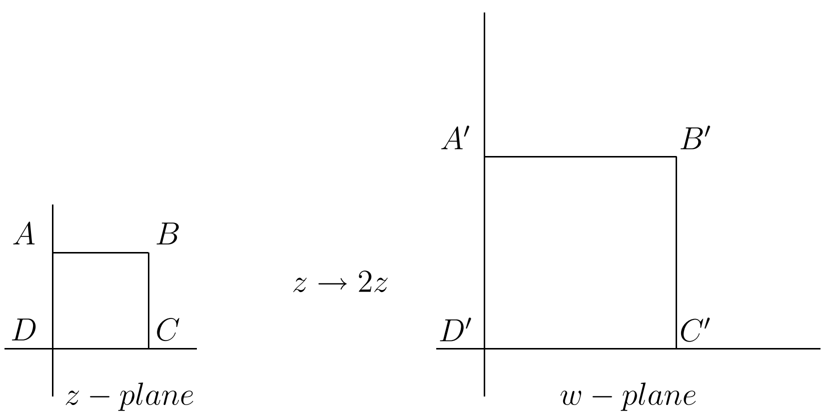


3) $w = \overline{z} + 2$, this is not a translation.



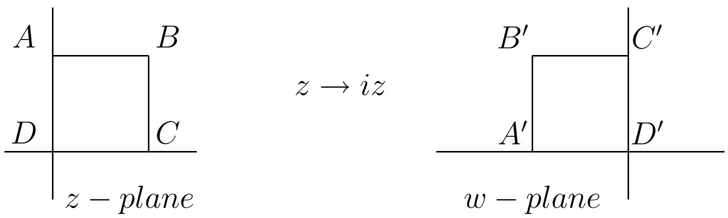
4) $w = 2z$ Now $|w| = 2|z|$ $\arg w = \arg 2 + \arg z = \arg z$

So this is an enlargement about the origin with scale factor 2.



5) $w = iz \quad |w| = |z| \quad \arg w = \arg i + \arg z = \dfrac{\pi}{2} + \arg z$

So this is a rotation through $\frac{\pi}{2}$ anticlockwise about O.



In general if α is any complex number and we write $\alpha = re^{i\theta}$ then $w = \alpha z$ is an enlargement by scale factor r together with a rotation about O through the angle θ anticlockwise.

If we write

$$\begin{aligned} \alpha &= a + ib \\ z &= x + iy \\ w &= u + iv \end{aligned}$$

then $w = \alpha z$

becomes $u + iv = (a + ib)(x + iy)$

and so

$$\begin{aligned} u &= ax - by \\ v &= bx + ay \end{aligned}$$

We write this in the form $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

The right hand side can be interpreted as a multiplication, but at the moment it seems a rather odd kind of multiplication.

We call $\begin{pmatrix} x \\ y \end{pmatrix}$ a column vector.

We call $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ a matrix.

If we now have another transformation $\xi = \beta w$ where $\beta = c + id$ then if we write $\xi = s + it$ we shall have

$$\begin{aligned} \begin{pmatrix} s \\ t \end{pmatrix} &= \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

If we now do the substitutions

$$s = cu - dv$$
$$t = du + cv$$

in the first pair of equations we get

$$s = (ca - db)x - (cb + da)y$$
$$t = (ad + bc)x + (ac - bd)y$$

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} (ca - bd) & -(cb + da) \\ (ad + bc) & (ac - bd) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This suggests that we should define

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} (ca - bd) & -(cb + da) \\ (ad + bc) & (ac - bd) \end{pmatrix}$$

Finally if we go back to the original equation $w = \alpha z \;\; v = \beta w$ we obtain $\xi = \beta\alpha z$ and $\beta\alpha = (c + id)(a + ib) = (ac - bd) + i(ad + bc)$

If we write α and β in polar form, taking $r = 1$ for both, so that they both correspond to rotations, we then have

$$\alpha = \cos\theta + i\sin\theta$$
$$\beta = \cos\phi + i\sin\phi$$

The corresponding matrices are

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -(\cos\theta\sin\phi + \sin\theta\cos\phi) \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix}$$

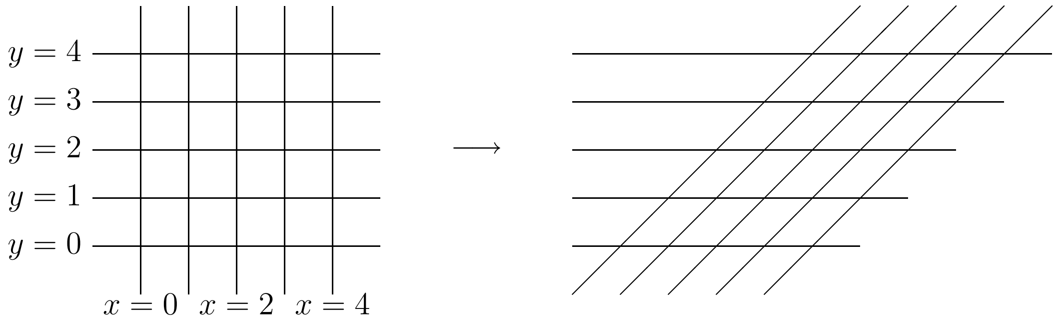
$$= \begin{pmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) \\ \sin(\theta + \phi) & \cos(\theta + \phi) \end{pmatrix}$$

which is in accordance with what we found previously.

Notice that although each complex number can be represented by a matrix, matrices such as $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ do not correspond to complex numbers. We can nevertheless use them to transform the plane.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y \end{pmatrix}$$

This corresponds to a shearing transformation.



In considering matrices used as transformations we have so far considered the problem of finding the image of given points.

$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$

i.e. given $\begin{pmatrix} x \\ y \end{pmatrix}$ what is $\begin{pmatrix} X \\ Y \end{pmatrix}$?

We now consider the reverse problem:

given $\begin{pmatrix} X \\ Y \end{pmatrix}$ what is $\begin{pmatrix} x \\ y \end{pmatrix}$?

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$

so

$$\begin{aligned} ax + by &= X \quad (1) \\ cx + dy &= Y \quad (2) \end{aligned}$$

$(1) * d$ and $(2) * b \Rightarrow$

$$\begin{aligned} adx + bdy &= dX \\ bcx + bdy &= bY \end{aligned}$$

subtracting gives

$$(ad - bc)x = dX - bY \quad (3)$$

$(1) * c$ and $(2) * a \Rightarrow$

$$\begin{aligned} acx + bcy &= cX \\ acx + ady &= aY \end{aligned}$$

subtracting gives

$$(ad - bc)y = aY - cX \quad (4)$$

(3) and (4) can be solved for x and y iff $ad - bc \neq 0$. If $ad - bc \neq 0$ we then have

$$x = \frac{d}{ad - bc}X - \frac{b}{ad - bc}Y$$

$$y = \frac{-c}{ad - bc}X + \frac{a}{ad - bc}Y$$

so

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$
$$= \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$
$$= \frac{1}{\triangle} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The matrix
$$\begin{pmatrix} \frac{d}{\triangle} & \frac{-b}{\triangle} \\ \frac{-c}{\triangle} & \frac{a}{\triangle} \end{pmatrix}$$
is called the inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ written A^{-1}

$A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

As a transformation this matrix does nothing at all. All points are fixed. It is called the identity matrix.

$\triangle = ad - bc$ is called the determinant of A. So A has an inverse iff its determinant is non-zero.

For a complex number matrix

$\alpha = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ $\triangle = a^2 + b^2 = |\alpha|^2$
 $\triangle = 0$ iff $a = b = 0$ i.e. $\alpha = 0$

and its inverse is

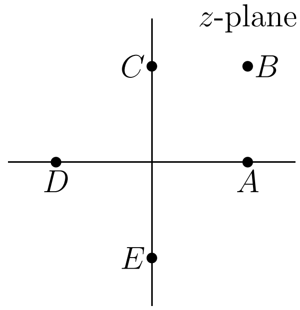
$$\frac{1}{|\alpha|^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \frac{\overline{\alpha}}{|\alpha|^2} = \frac{1}{\alpha} \qquad \alpha \neq 0$$

In widening the system to include all possible 2×2 matrices we have included many matrices which do not have inverses. We have also sacrificed commutativity of multiplication, as AB does not always equal BA.

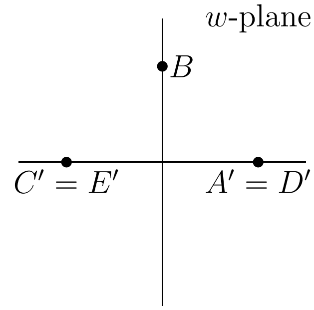
However we can deal with many different transformations, and matrices turn out to have many and varied applications.

Other transformations

There are many transformations not represented by 2×2 matrices as above. As an example we consider a few properties of the transformation $w = z^2$. It is convenient to use polar co-ordinates, we use (r, θ) in the z-plane and (p, ϕ) in the w-plane.

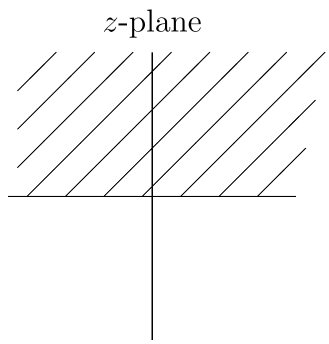


$$z = re^{i\theta} \to z^2 = r^2 e^{2i\theta}$$
$$\text{so } p = r^2$$
$$\phi = 2\theta$$

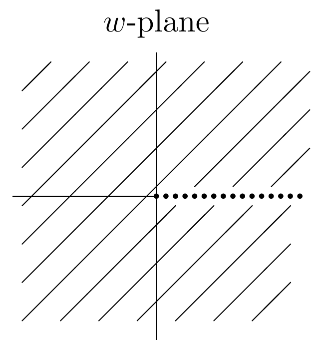


$A(1, 0)$	$A'(1, 0)$
$B(\sqrt{2}, \frac{\pi}{4})$	$B'(2, \frac{\pi}{2}$
$C(1, \frac{pi}{2}$	$C'(1, \pi)$
$D(1, \pi)$	$D'(1, 2\pi) = (1, 0)$
$E(1, \frac{3\pi}{2}$	$E'(1, 3\pi) = (1, \pi)$

DIAGRAM
so $z = e^{i\theta} \quad -\pi \leq \theta \leq \pi$ corresponds to a circle traced twice in the w-plane.
DIAGRAM



$$\text{so } z = re^{i\theta} \ 0 < \theta < \pi$$
$$w = r^2 e^{2i\theta} \ 0 < 2\theta < 2\pi$$
$$= pe^{i\theta} \ 0 < \phi < 2\pi$$



upper half plane $y > 0$

plane without +ve real axis

Reverting to cartesians now let $z = x + iy \quad w = \xi + i\eta$
$\xi + i\eta = x^2 - y^2 + 2ixy$ so $\xi = x^2 - y^2 \quad \eta = 2xy$
Now if $x = 1, \quad \xi = 1 - y^2 \quad \eta = 2y$
so $\xi = 1 - \dfrac{\eta^2}{4}$
DIAGRAM
If $y = 1 \quad \xi = x^2 - 1 \quad \eta = 2x$ so $\xi = \dfrac{\eta^2}{4} - 1$
DIAGRAMS